# Koszul duality and categorical dynamics

Li, Yin

*Awarding institution:*
King's College London

# Koszul duality and categorical dynamics

Yin Li

A thesis presented for the degree of

Doctor of Philosophy

Department of Mathematics

King's College London

May 2020

# Abstract

This thesis is essentially a combination of my papers [74] and [75]. We study two closely related topics in symplectic topology: Koszul duality and categorical dynamics.

Koszul duality is a duality phenomenon between homologically smooth and proper $A_\infty$-algebras. When applied in the geometric context, it requires that the wrapped Fukaya category $\mathcal{W}(M)$ of a Liouville manifold to be recoverable from its full subcategory $\mathcal{F}(M)$ of compact Lagrangian submanifolds, which in particular implies the isomorphism $HH^*(\mathcal{W}(M)) \cong HH^*(\mathcal{F}(M))$ between the Hochschild cohomologies. Categorical dynamics, on the other hand, studies (infinitesimal) symmetries on the Fukaya category $\mathcal{F}(M)$. When such symmetries have geometric origins, they define cohomology classes in the symplectic cohomology $SH^*(M)$, i.e. Hochschild cohomology $HH^*(\mathcal{W}(M))$ of the wrapped Fukaya category. In particular, the restriction morphism $HH^*(\mathcal{W}(M)) \to HH^*(\mathcal{F}(M))$ is non-vanishing on the classes defining the symmetries on $\mathcal{F}(M)$. Thus both of the theories can be regarded as imposing restrictions on the symplectic topology of $M$ so that the Fukaya categories $\mathcal{F}(M)$ and $\mathcal{W}(M)$ are closely related.

To explore the relationships between these two theories, we show that the method of constructing Liouville manifolds whose Fukaya categories $\mathcal{F}(M)$ carry dilating $\mathbb{C}^*$-

actions can also be applied to obtain examples of Liouville manifolds whose Fukaya categories are related by $A_\infty$-Koszul duality. On the other hand, we show that for many Liouville manifolds whose Fukaya categories are Koszul dual, there is a generalized theory of categorical dynamics, so that infinitesimal symmetries exist on $A_\infty$-subcategory of $\mathcal{F}(M)$ consisting of mutually orthogonal spherical objects, and these infinitesimal symmetries can be patched together to give rise to a cyclic homology class in $HC_{-n+1}(\mathcal{W}(M))$. This is the so-called exact Calabi-Yau structure to be mentioned below. As a consequence, this thesis is naturally divided into two parts.

In the first part of this thesis (Chapters 3, 4, 5, 6), we consider a family of 6-dimensional Milnor fibers $M_{p,q,r}$ which are affine hypersurfaces in $\mathbb{C}^4$. They are Milnor fibers of stabilizations of cusp and simple elliptic singularities in $\mathbb{C}^3$. Explicit computations enable us to identify their compact Fukaya categories $\mathcal{F}(M_{p,q,r})$ with the cyclic completions of certain directed quiver algebras $\mathcal{A}_{p,q,r}$, and their wrapped Fukaya categories $\mathcal{W}(M_{p,q,r})$ with the Calabi-Yau completions of the same quiver algebras $\mathcal{A}_{p,q,r}$, therefore showing that the Fukaya categories $\mathcal{F}(M_{p,q,r})$ and $\mathcal{W}(M_{p,q,r})$ are related to each other via $A_\infty$-Koszul duality. As applications, we prove the formality of the Fukaya $A_\infty$-algebra of a basis of vanishing cycles in $M_{p,q,r}$, and show that the compact Fukaya category $\mathcal{F}(M_{p,q,r})$ is split-generated by vanishing cycles. In particular, the manifolds $M_{p,q,r}$ provide interesting examples of Liouville manifolds whose wrapped Fukaya categories are exact Calabi-Yau in the sense of Keller.

An exact Calabi-Yau structure on a homologically smooth $A_\infty$-category, being the key notion for our study in the second part of this thesis (Chapters 7, 8, 9), is a special kind of smooth Calabi-Yau structures in the sense of Kontsevich-Vlassopoulos [69]. For a Weinstein manifold $M$, the existence of an exact Calabi-Yau structure on its wrapped Fukaya category $\mathcal{W}(M)$ imposes strong restrictions on its symplectic topology. Under the cyclic open-closed map constructed by Ganatra [43], an exact Calabi-Yau structure on $\mathcal{W}(M)$ induces a class $\tilde{b}$ in the degree one equivariant symplectic cohomology $SH^1_{S^1}(M)$. Using $\tilde{b} \in SH^1_{S^1}(M)$, we construct an endomorphism on the Floer cohomology $HF^*(L, L)$ of a Lagrangian sphere $L \subset M$ with dimension $n \geqslant 3$, which acts trivially on $HF^0(L, L)$, and non-trivially on $HF^n(L, L)$. This enables us to prove that for many Weinstein manifolds with exact Calabi-Yau wrapped Fukaya categories, there is an upper bound on the number of disjoint Lagrangian spheres, and the rational homology classes of these spheres are linearly independent. These results generalize those of Seidel in [96] since any Weinstein manifold admitting a quasi-dilation has an exact Calabi-Yau wrapped Fukaya category. Finally, using Koszul duality, we prove that there are examples of Weinstein manifolds whose wrapped Fukaya categories are exact Calabi-Yau, but which do not admit quasi-dilations.

# Dedication

*To Hong Kong's protesters, with sincere affection.*

# Acknowledgements

I am indebted to my thesis advisor, Yankı Lekili, for his patient guidance and numerous useful suggestions during the preparation of this thesis. I have benefited very much from his encouragements and the inspiring discussions that we had during the last more than three years. I am also grateful to the tremendous effort he has put in publicizing my work and introducing me to many good mathematicians, which enables me to worry less about social relations and focus exclusively on mathematics.

I would like to thank Professors Tobias Ekholm, Paul Seidel and Ivan Smith for their interests in my work, and for many useful conversations that I had with them during my visits. I am indebted to Sheel Ganatra, Ailsa Keating, Nick Sheridan and Zhengyi Zhou for pointing out numerous mistakes in earlier versions of this thesis. Discussions with Roger Casals, Daniel Pomerleano, Travis Schedler and Jingyu Zhao turn out to be fruitful and contributed at many places to this thesis. I should also thank Guangbo Xu, who invited me to visit Princeton University in April, 2018, during which I presented the results in the first part of this thesis.

I would also like to thank Jonny Evans, with whose warm encouragements I went thorough the most difficult times during my PhD studies. Also appreciated are the help of my friends in London School of Geometry and Number Theory (LSGNT), Momchil Konstantinov, Kristina Kubiliūtė, and Emily Maw after my traffic accident in April, 2017.

I have also learned a lot through conversations with many graduate students, postdocs and mathematicians at different places during the last few years. To name a few, I have benefited from the conversations with Doğancan Karabaş during my study at London, with Georgios Dimitroglou-Rizell, Luis Diogo, Yang Huang, Thomas Kragh and Maksim Maydanskiy when visiting Uppsala, and Mark McLean during a workshop at Jersey.

I would also like to express my gratitude to Ying for her consistent support and unconditional love.

Finally, since it is difficult for me to recall everyone who helped me during these years, I'll give thanks to God.

My doctoral studies were kindly funded by the Engineering and Physical Sciences Research Council [EP/L015234/1], the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

# Contents

# List of Figures

# Chapter 1

# Introduction

We will work throughout this thesis with a field $\mathbb{K}$, whose algebraic closure is denoted by $\overline{\mathbb{K}}$. All the dg or $A_\infty$-categories in this thesis will be defined over $\mathbb{K}$, so do the corresponding homotopy or homology theories. When the categories are split-generated by finitely many objects in the sense of [89], it is convenient to use an equivalent language, namely dg or $A_\infty$-algebras over the semisimple ring $\Bbbk := \bigoplus_{i\in I} \mathbb{K}e_i$, where $I$ is a finite set and $\{e_i\}_{i\in I}$ is a set of idempotents indexed by $I$. In this way, we are not going to distinguish below between an $A_\infty$-category $\mathcal{A}$ split-generated by finitely many objects $\{S_i\}_{i\in I}$ and its endomorphism algebra of the object $\bigoplus_{i\in I} S_i$ in the formal enlargement $\mathcal{A}^{tw}$, which is an $A_\infty$-algebra over $\Bbbk$.

All the dg or $A_\infty$-algebras in this thesis will be $\mathbb{Z}$-graded. The Hochschild chain complex of an $A_\infty$-algebra $\mathcal{A}$ will be denoted by $CH_*(\mathcal{A})$, and we use $HH_*(\mathcal{A})$ to denote its homology. The more familiar notation $CC_*(\mathcal{A})$ will be reserved for the cyclic chain complex, which computes the cyclic homology $HC_*(\mathcal{A})$. The same notational convention will be adopted when dealing with Hochschild and cyclic cochains, with the subscripts indicating the gradings replaced by superscripts.

## 1.1 Koszul duality in symplectic topology

Let $M$ be a $2n$-dimensional Liouville manifold, which can be regarded as the completion

$$M = \overline{M} \cup_{\partial \overline{M}} \mathbb{R}_{\geqslant 0} \times \partial \overline{M} \tag{1.1}$$

of a Liouville domain $\overline{M}$, where $\partial \overline{M}$ is a contact manifold with the contact form given by the restriction of the Liouville form $\theta_M$. For any field $\mathbb{K}$, we can associate to $M$ two versions of Fukaya categories: the *compact Fukaya category* $\mathcal{F}(M)$ and the *wrapped Fukaya category* $\mathcal{W}(M)$. Under the assumption that $c_1(M) = 0$, they are $\mathbb{Z}$-graded $A_\infty$-categories over $\mathbb{K}$.

Recall that the objects of $\mathcal{F}(M)$ are oriented, *Spin*, closed exact Lagrangian submanifolds $L \subset M$ with vanishing Maslov class (when $\mathrm{char}(\mathbb{K}) = 2$, the orientable and *Spin* assumptions on $L$ can be removed). Let $L_1, \ldots, L_r$ be a finite collection of objects in $\mathcal{F}(M)$, using compactly supported Hamiltonian perturbations, one can always achieve that these Lagrangian submanifolds are intersecting transversally, so that the Floer cochain complexes $CF^*(L_i, L_j)$ are well-defined. The $A_\infty$-relations on the chain level are established rigorously by Seidel in [89], which makes $\mathcal{F}(M)$ a well-defined $A_\infty$-category.

Since $M$ is non-compact, it is also natural to take into account certain non-compact Lagrangian submanifolds of $M$, and define an $A_\infty$-category $\mathcal{W}(M)$ which has possibly infinite dimensional morphism spaces given by the wrapped Floer cochain complexes $CW^*(L_i, L_j)$. To be concrete, the non-compact Lagrangian submanifolds which define objects of $\mathcal{W}(M)$ are those modelled on a cone $\mathbb{R}_{\geqslant 0} \times \Lambda$ over the cylindrical end $\mathbb{R}_{\geqslant 0} \times \partial \overline{M}$, where $\Lambda \subset \partial \overline{M}$ is a closed Legendrian submanifold. The $A_\infty$-structure on $\mathcal{W}(M)$ can be defined either by using a Hamiltonian function which is linear on the cylindrical end, together with the telescope construction [7] or a

Hamiltonian function which is quadratic on the cylindrical end, plus some appropriate rescalings of the Liouville flow [2].

As a typical example, consider the situation when $M = T^*Q$ is the cotangent bundle of a compact smooth manifold $Q$, the $A_\infty$-categories $\mathcal{F}(M)$ and $\mathcal{W}(M)$ have topological interpretations, see for example [3, 4]. Denote by $\mathcal{F}_M$ and $\mathcal{W}_M$ the Fukaya $A_\infty$-algebras of the zero-section $CF^*(Q,Q)$ and the fiber $CW^*(T_q^*Q, T_q^*Q)$ respectively, we have the following quasi-isomorphisms:

$$\mathcal{F}_M \cong C^*(Q;\mathbb{K}); \mathcal{W}_M \cong C_{-*}(\Omega_q Q;\mathbb{K}), \tag{1.2}$$

where $C^*(Q;\mathbb{K})$ is the dg algebra of singular cochains on $Q$ and $C_{-*}(\Omega_q Q;\mathbb{K})$ is the dg algebra of chains on the based loop space $\Omega_q Q$.

A useful tool of studying the dg algebras $C^*(Q;\mathbb{K})$ and $C_{-*}(\Omega_q Q;\mathbb{K})$ is *Adams' cobar construction* [9, 10]. Recall that there are natural augmentations

$$\varepsilon_{\mathcal{F}} : C^*(Q;\mathbb{K}) \to \mathbb{K}, \varepsilon_{\mathcal{W}} : C_{-*}(\Omega_q Q;\mathbb{K}) \to \mathbb{K}, \tag{1.3}$$

which endow $C^*(Q;\mathbb{K})$ and $C_{-*}(\Omega_q Q;\mathbb{K})$ with the structures of augmented dg algebras, where $\varepsilon_{\mathcal{F}}$ is induced by the inclusion $pt \hookrightarrow Q$ of a point, and $\varepsilon_{\mathcal{W}}$ comes from the trivial local system $\pi_1(Q, pt) \to \mathbb{K}$. It follows from the *Eilenberg-Moore equivalence* that

$$R\mathrm{Hom}_{C_{-*}(\Omega_q Q;\mathbb{K})}(\mathbb{K},\mathbb{K}) \cong C^*(Q;\mathbb{K}). \tag{1.4}$$

If we further assume that $Q$ is *simply-connected*, it follows from Adams' cobar construction that there is another quasi-isomorphism

$$R\mathrm{Hom}_{C^*(Q;\mathbb{K})}(\mathbb{K},\mathbb{K}) \cong C_{-*}(\Omega_q Q;\mathbb{K}), \tag{1.5}$$

namely $C^*(Q;\mathbb{K})$ and $C_{-*}(\Omega_q Q;\mathbb{K})$ are Koszul dual as augmented dg algebras.

Generalizations of the above Koszul duality in the context of symplectic topology have been obtained by Etgü-Lekili [40] and Ekholm-Lekili [36]. More specifically, they considered the case when $M$ is a plumbing of cotangent bundles $T^*Q_v$ of simply connected manifolds $Q_v$ according a tree $T = (T_0, T_1)$, where $T_0$ is the set of vertices, and $T_1$ is the set of edges. Denote by

$$\mathcal{F}_M := \bigoplus_{v,w \in T_0} CF^*(Q_v, Q_w) \tag{1.6}$$

the endomorphism algebra of the zero sections $Q_v \subset T^*Q_v, Q_w \subset T^*Q_w$ in the compact Fukaya category $\mathcal{F}(M)$, and by

$$\mathcal{W}_M := \bigoplus_{v,w \in T_0} CW^*(L_v, L_w) \tag{1.7}$$

the endomorphism algebra of cotangent fibers $L_v = T_q^*Q_v, L_w = T_q^*Q_w$ in the wrapped Fukaya category $\mathcal{W}(M)$. Up to quasi-isomorphism, these are strictly unital $A_\infty$-algebras over the semisimple ring $\Bbbk := \bigoplus_{v \in T_0} \mathbb{K} e_v$, where $e_v$ is an idempotent in $CF^0(Q_v, Q_v)$ or $CW^0(L_v, L_v)$.

As above, the $A_\infty$-algebras $\mathcal{F}_M$ and $\mathcal{W}_M$ are equipped with augmentations

$$\varepsilon_{\mathcal{F}} : \mathcal{F}_M \to \Bbbk, \varepsilon_{\mathcal{W}} : \mathcal{W}_M \to \Bbbk, \tag{1.8}$$

where $\varepsilon_{\mathcal{F}}$ is defined by projecting to $\Bbbk \subset \mathcal{F}_M^0$, while $\varepsilon_{\mathcal{W}}$ is induced from the exact Lagrangian filling $\left(\bigcup_{v \in T_0} Q_v\right) \cap D^{2n}$ of the Legendrian submanifold $\left(\bigcup_{v \in T_0} Q_v\right) \cap \partial D^{2n} \subset (S^{2n-1}, \xi_{std})$. In [36, 40] it is proved that there are quasi-isomorphisms

$$R\mathrm{Hom}_{\mathcal{W}_M}(\Bbbk, \Bbbk) \cong \mathcal{F}_M, R\mathrm{Hom}_{\mathcal{F}_M}(\Bbbk, \Bbbk) \cong \mathcal{W}_M \tag{1.9}$$

when

- $\dim_{\mathbb{R}}(M) = 4$ and $M$ is a plumbing of $T^*S^2$'s with $T = A_n$ or $D_n$ and $\mathrm{char}(\mathbb{K}) \neq 2$;

- $\dim_{\mathbb{R}}(M) \geqslant 6$ and $M$ is a plumbing of $T^*Q_v$'s according to any tree $T$, with each $Q_v$ being simply-connected.

**Conventions**

- In this thesis, we will need to deal with, at certain points, a bigraded dg or $A_\infty$-algebra $\mathcal{A}$ over $\Bbbk$, namely $\mathcal{A} = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{A}^{i,j}$ as a $\Bbbk$-bimodule, and the differential or $A_\infty$-operations changes only the first grading $i$, see Section 2.5 for details. In general, the Koszul dual $R\mathrm{Hom}_{\mathcal{A}}(\Bbbk, \Bbbk)$ of $\mathcal{A}$ (where $R\mathrm{Hom}_{\mathcal{A}}$ is taken in the derived category of bigraded $\mathcal{A}$-modules), regarded as a singly graded $A_\infty$-algebra with respect to the total grading $i + j$, differs from the Koszul dual of $\mathcal{A}$ regarded as a bigraded $A_\infty$-algebra. As an example, consider $\mathcal{A} = \mathbb{K}[x]/(x^2)$ with $|x| = (0, 1)$. In the first case, $R\mathrm{Hom}_{\mathcal{A}}(\Bbbk, \Bbbk)$ is isomorphic to the ring of formal power series $\mathbb{K}[[y]]$ with $|y| = 0$, while in the second case one gets the polynomial ring $\mathbb{K}[y]$ with $|y| = (1, -1)$. In order to distinguish between these two situations, we will use the notation $\mathcal{A}^!$ for the *singly graded* Koszul dual of $\mathcal{A}$, and use $E(\mathcal{A})$ to stand for the *bigraded* Koszul dual.

- We will always regard $\Bbbk$ as a left $\mathcal{F}_M$-module, and a right $\mathcal{W}_M$-module. One can equivalently view $\Bbbk$ as a right $\mathcal{F}_M$-module, so that in the second formula of (1.9) above becomes $R\mathrm{Hom}_{\mathcal{F}_M^{op}}(\Bbbk, \Bbbk) \cong \mathcal{W}_M^{op}$, where $\mathcal{A}^{op}$ is the opposite of $\mathcal{A}$.

More generally, for any Weinstein manifold $M$, and a fixed handlebody decomposition of $M$, we can consider the Lagrangian cocores $L_1, \cdots, L_r$ of the critical handles and form the endomorphism algebra $\mathcal{W}_M$ in the wrapped Fukaya category $\mathcal{W}(M)$. According to [23] and [47], the $A_\infty$-algebra $\mathcal{W}_M$ is independent of the choice of the handlebody decomposition up to Morita equivalence, and we will call it the *wrapped Fukaya $A_\infty$-algebra.*

In concrete geometric situations, for instance, when $M$ is the Milnor fiber associated to a quasi-homogeneous singularity, it is usually possible to find a corresponding set of objects $V_1, \cdots, V_r$ of $\mathcal{F}(M)$ which is "dual" to $L_1, \cdots, L_r$ in the sense that $V_i \cap L_j = \varnothing$ if $i \neq j$, and $V_i$ intersects $L_i$ transversely at a unique point. Denote by $\mathcal{V}_M$ the endomorphism algebra of $V_1, \cdots, V_r \subset M$ in $\mathcal{F}(M)$, which we will refer to as the *compact Fukaya $A_\infty$-algebra*. Note that both $\mathcal{V}_M$ and $\mathcal{W}_M$ are $A_\infty$-algebras over $\mathbb{k} = \bigoplus_{1 \leqslant i \leqslant r} \mathbb{K} e_i$. Now it is natural to ask the following question:

**Question 1.1.1.** *When are the $A_\infty$-algebras $\mathcal{V}_M$ and $\mathcal{W}_M$ Koszul dual in the sense of (1.9)?*

Note that the Liouville manifolds which satisfy the Koszul duality (1.9) form only a very restrictive class. Known counterexamples include cotangent bundles of most of the non-simply connected manifolds and their plumbings, plumbings of $T^*S^2$'s according to a non-Dynkin tree [41], and many smooth log Calabi-Yau varieties [61, 109].

From a more speculative perspective, when $M$ is a smooth affine variety, the answer to Question 1.1.1 is related to the notion of *log Kodaira dimension*

$$\kappa(M) \in \{-\infty\} \cup \{0, \cdots, n\}. \tag{1.10}$$

This is defined by choosing a compactification $X$ of $M$ so that $X$ is a smooth projective variety, and the divisor $D = X \backslash M$ is simple normal crossing. $\kappa(M)$ is defined as the Kodaira-Iitaka dimension of the line bundle $K_X + D$ over $X$. In fact, assume that $\mathcal{F}(M)$ is split-generated by the Lagrangians $V_1, \cdots, V_r$, it follows from Koszul duality and the property of the closed-open string map [44] that

$$SH^0(M) \cong HH^0(\mathcal{W}(M)) \cong HH^0(\mathcal{F}(M)), \tag{1.11}$$

which suggests that the zeroth symplectic cohomology $SH^0(M)$ should be finite dimensional [1]. When $D \subset X$ is a Donaldson hypersurface with $[D] = m \cdot c_1(X)$, this is precisely the case when $m \in \mathbb{Q}_{>0}$ and $m \neq 1$, which suggests the following:

**Conjecture 1.1.1.** *Let $M$ be an $n$-dimensional smooth affine variety, so that one can find a set of closed exact Lagrangian submanifolds to form the $A_\infty$-algebra $\mathcal{V}_M$. If $\mathcal{V}_M$ and $\mathcal{W}_M$ are Koszul dual as $A_\infty$-algebras over $\mathbb{k}$, then*

$$\kappa(M) = -\infty \textit{ or } \kappa(M) = n. \tag{1.12}$$

*If we further assume that $M$ is a Milnor fiber associated to an isolated singularity, then the condition (1.12) should be sufficient.*

There is a systematic way of constructing Milnor fibers with $\kappa(M) = -\infty$, which is known as *stabilization*. In its general form, the construction can be described as follows. Let

$$\pi : E \to \mathbb{C} \tag{1.13}$$

be an exact symplectic Lefschetz fibration on some $(2n-2)$-dimensional Liouville manifold $E$ with its smooth fiber given by the Liouville manifold $M$, where $n > 2$. Associated to $\pi$ is a directed $A_\infty$-category $\mathcal{A}(\pi)$ over $\mathbb{K}$, whose objects are objects of the compact Fukaya category $\mathcal{F}(E)$, together with the Lefschetz thimbles of $\pi$, and the morphism space $CF^*(\Delta_i, \Delta_j)$ between two thimbles $\Delta_i$ and $\Delta_j$, if non-trivial, is defined using a Hamiltonian perturbation which is small at infinity, see [89]. Consider

[1] There is in general no reason to believe that $HH^0$ of a proper $A_\infty$-algebra is finite dimensional. However, looking at known examples it seems that if $\mathcal{F}_M$ and $\mathcal{W}_M$ are Koszul dual, the mirror of $M$ should either be a locally finite Ginzburg dg algebra, therefore non-commutative, or a proper Calabi-Yau variety. In both of the cases, $HH^0$ is finite-dimensional.

the *suspension* of $\pi$, which is the Lefschetz fibration

$$\pi^\sigma : E \times \mathbb{C} \to \mathbb{C} \tag{1.14}$$

defined by $\pi + w^2$, where $w$ is the holomorphic coordinate on the factor $\mathbb{C}$. A smooth fiber of $\pi^\sigma$ is a Liouville manifold $M^\sigma$. Denote by

$$\mathcal{V}(M^\sigma) \subset \mathcal{F}(M^\sigma) \tag{1.15}$$

the full $A_\infty$-subcategory formed by the Lagrangian spheres $V_i^\sigma \subset M^\sigma$ which are double branched covers of the Lefschetz thimbles $\Delta_i \subset E$ of $\pi$. Applying the suspension construction once more we obtain a Lefschetz fibration $\pi^{\sigma\sigma} : E \times \mathbb{C}^2 \to \mathbb{C}$. It is proved by Seidel in [92] that we have the following quasi-isomorphism between $A_\infty$-categories

$$\mathcal{V}(M^{\sigma\sigma}) \cong \mathcal{A}(\pi) \oplus \mathcal{A}(\pi)^\vee[-n], \tag{1.16}$$

where the $A_\infty$-structure on the right-hand side is the trivial extension of that on $\mathcal{A}(\pi)$, see Section 3.1. Equivalently, (1.16) can be expressed in terms of endomorphism algebras, namely $\mathcal{V}_{M^{\sigma\sigma}} \cong \mathcal{A}_\pi \oplus \mathcal{A}_\pi^\vee[-n]$, where

$$\mathcal{V}_{M^{\sigma\sigma}} := \bigoplus_{i,j} CF^*(V_i^{\sigma\sigma}, V_j^{\sigma\sigma}) \text{ and } \mathcal{A}_\pi := \bigoplus_{i,j} CF^*(\Delta_i, \Delta_j) \tag{1.17}$$

are $A_\infty$-algebras over $\mathbb{k}$.

In the special case when $E \cong \mathbb{C}^{n-1}$, we necessarily have $\kappa(M^{\sigma\sigma}) = -\infty$. In view of Conjecture 1.1.1, it is therefore natural to expect that the Fukaya $A_\infty$-algebras of double stabilizations of Milnor fibers are related by $A_\infty$-Koszul duality.

In algebraic terms, the $A_\infty$-algebra $\mathcal{V}_{M^{\sigma\sigma}}$ is the *cyclic completion* of the directed $A_\infty$-algebra $\mathcal{A}_\pi$ in the sense of Segal [88]. There is also a Koszul dual construction, due to Keller [62], called *Calabi-Yau completion*, see Section 2.4. When applied to

$\mathcal{A}_\pi$, it produces a smooth $n$-Calabi-Yau algebra $\Pi_n(\mathcal{A}_\pi)$ in the sense of Ginzburg [50], see Section 1.3. Therefore, to prove Koszul duality between the $A_\infty$-algebras $\mathcal{V}_{M^{\sigma\sigma}}$ and $\mathcal{W}_{M^{\sigma\sigma}}$, it suffices to identify the wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_{M^{\sigma\sigma}}$ with the $n$-Calabi-Yau completion $\Pi_n(\mathcal{A}_\pi)$. As remarked above, this has already been achieved in the case when $M^{\sigma\sigma}$ is the Milnor fiber associated to a simple singularity and $\dim_{\mathbb{R}}(M^{\sigma\sigma}) \geqslant 6$, in which case $\Pi_n(\mathcal{A}_\pi)$ is the Ginzburg algebra associated to a quiver with trivial potential. In the next section we will see that there are interesting examples of Liouville manifolds whose wrapped Fukaya categories are described by quivers with non-trivial potentials.

At this point, it seems to be appropriate to mention that suspending Lefschetz fibrations is also an effective way of obtaining Liouville manifolds whose Fukaya categories $\mathcal{F}(M)$ admitting dilating $\mathbb{C}^*$-actions, see Lecture 16 of [90] for details. We will return to the theory of categorical dynamics in Section 1.5.

## 1.2 Stabilizations of hypersurface cusp singularities

Consider the symplectic 6-manifolds $M_{p,q,r} \subset \mathbb{C}^4$ which are Milnor fibers associated to the isolated singularities

$$x^p + y^q + z^r + \lambda xyz + w^2 = 0, \tag{1.18}$$

where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leqslant 1. \tag{1.19}$$

The constant $\lambda \in \mathbb{C}$ above is allowed to take all but finitely many values, see [60] for details. To be explicit, we will take in this thesis $\lambda = 1$ when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, and

$\lambda = 0$ when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Changing the value of $\lambda$ will not change the symplectic structure on $M_{p,q,r}$ up to exact symplectomorphism.

Note that the above manifolds are stabilizations of the Milnor fibers $T_{p,q,r} \subset \mathbb{C}^3$ associated to the non-simple isolated singularity of modality one

$$t_{p,q,r}(x, y, z) := x^p + y^q + z^r + \lambda xyz = 0, \tag{1.20}$$

which are studied by Keating in [60] and [61]. Equivalently, they can be realized as smooth fibers of the suspension of the Lefschetz fibration $\tilde{t}_{p,q,r} : \mathbb{C}^3 \to \mathbb{C}$, with $\tilde{t}_{p,q,r}$ being a Morsification of $t_{p,q,r}$. The endomorphism algebra of the directed $A_\infty$-category $\mathcal{A}(\tilde{t}_{p,q,r})$ will be denoted by $\mathcal{A}_{p,q,r}$.

In fact, Theorem 1.2.1 below and its applications hold in the more general case when $p \geqslant 2, q \geqslant 2$ and $r \geqslant 2$, and we can always take $\lambda = 1$ for these triples $(p, q, r) = (k, 2, 2), (3, 3, 2), (4, 3, 2), (5, 3, 2)$, where $k \geqslant 2$. In these cases, the corresponding Weinstein manifolds $M_{p,q,r}$ are no longer Milnor fibers associated to any isolated singularity, but still share a lot of properties similar to Milnor fibers, e.g. they are homotopy equivalent to wedges of Lagrangian spheres. For this reason, these Weinstein manifolds are called *generalized Milnor fibers*, see [60] for a detailed discussion of their symplectic topology.

Unless otherwise stated, we shall impose $p \geqslant 2, q \geqslant 2, r \geqslant 2$ as our standing assumptions when referring to the Weinstein manifolds $M_{p,q,r}$. In particular, this applies to Theorem 1.2.1 and its applications (Corollaries 1.2.1 to 1.2.3). In fact, $M_{p,q,r}$ is a well-define Stein manifold whenever $p, q, r$ are non-negative, and Koszul duality holds when $p \geqslant 1, q \geqslant 1, r \geqslant 1$. See also Section 6.4 for a brief discussion for other triples $(p, q, r)$.

Before stating our theorem, recall that any Weinstein domain $\overline{M}$ can be constructed by attaching critical handles to a subcritical Weinstein domain $\overline{M}_0$ along a disjoint union of Legendrian spheres in $\partial\overline{M}_0$. In our case, it turns out that the handlebody decomposition of $\overline{M}_{p,q,r}$ is pretty simple in the sense that there are no subcritical handles involved (cf. Section 4.3), so that we can take $\overline{M}_0 = D^6$ and apply Legendrian surgery along the link $\Lambda_{p,q,r} \subset (S^5, \xi_{std})$ in the standard contact 5-sphere, which is a disjoint union of standard unknotted $S^2$'s. Denote by $CE^*(\Lambda_{p,q,r})$ the *Chekanov-Eliashberg dg algebra* associated to $\Lambda_{p,q,r}$ (see Section 6.3 for its definition), and let

$$\mathcal{W}_{p,q,r} := \bigoplus_{1\leqslant i\leqslant p+q+r-1} CW^*(L_i, L_j) \tag{1.21}$$

be the wrapped Fukaya $A_\infty$-algebra of the Lagrangian cocores of the critical handles. It is announced in Bourgeois-Ekholm-Eliashberg [16] and proved in [33] that there is a quasi-isomorphism between the Chekanov-Eliashberg algebra of a Legendrian link and the wrapped Fukaya $A_\infty$-algebra of the corresponding Lagrangian cocores, which when adapted to our case implies the quasi-isomorphism:

$$\mathcal{W}_{p,q,r} \cong CE^*(\Lambda_{p,q,r}). \tag{1.22}$$

With the notations fixed as above, our main result in the first part of this thesis can be stated as follows.

**Theorem 1.2.1.** *Let $\mathbb{K}$ be any field, and set $\Bbbk := \bigoplus_{1\leqslant i\leqslant p+q+r-1} \mathbb{K}e_i$. There is a quasi-isomorphism between dg algebras over $\Bbbk$:*

$$CE^*(\Lambda_{p,q,r}) \cong \Pi_3(\mathcal{A}_{p,q,r}). \tag{1.23}$$

**Remark 1.2.1.** *The 3-Calabi-Yau completion $\Pi_3(\mathcal{A}_{p,q,r})$ is quasi-isomorphic to the Ginzburg dg algebra $\mathcal{G}_{p,q,r}$ associated to a quiver $Q_{p,q,r}$ with non-trivial potential $w_{p,q,r}$, see Section 2.4 for a precise description. As far as we know, this gives the first set of Liouville manifolds whose wrapped Fukaya categories can be identified with a Ginzburg dg algebra defined by a quiver with a non-trivial potential.*

It seems to be appropriate to say here a few words about the method we use to prove Theorem 1.2.1. In order to get the Legendrian frontal description of the Milnor fibers $M_{p,q,r}$, we start with a Lefschetz fibration on $T_{p,q,r}$ constructed by Keating in [61], whose smooth fiber is symplectomorphic to a 3-punctured torus. This fibration induces naturally a Lefschetz fibration on the stabilization $M_{p,q,r}$, whose smooth fiber is symplectomorphic to a 4-dimensional $D_4$ Milnor fiber. Using an algorithm due to Casals-Murphy [22], this Lefschetz fibration on $M_{p,q,r}$ can then be translated to produce a Legendrian frontal description of $M_{p,q,r}$ which involves both 2-handles and 3-handles. After a standard procedure of handle cancellations and Reidemeister moves, we can simplify the front diagram so as to obtain the attaching link $\Lambda_{p,q,r} \subset S^5$ for $M_{p,q,r}$. In order to compute the Chekanov-Eliashberg dg algebra $CE^*(\Lambda_{p,q,r})$, we use the cellular model introduced by Rutherford-Sullivan [85, 86] for Legendrian surfaces. This enables us to simplify the analysis of Morse flow trees and compute $CE^*(\Lambda_{p,q,r})$ over $\mathbb{Z}/2$. To get a computation of $CE^*(\Lambda_{p,q,r})$ over any field $\mathbb{K}$, we appeal to the result of Karlsson [58, 59] on the orientation data of Morse flow trees.

Theorem 1.2.1 can be applied to understand the symplectic topology of the manifolds $M_{p,q,r}$. Denote by

$$\mathcal{V}_{p,q,r} := \bigoplus_{1 \leqslant i \leqslant p+q+r-1} CF^*(V_i, V_j) \tag{1.24}$$

the endomorphism algebra in the compact Fukaya category $\mathcal{F}(M_{p,q,r})$ of a basis of vanishing cycles $V_1, \ldots, V_{p+q+r-1} \subset M_{p,q,r}$. To be specific, we assume that the basis of vanishing cycles is chosen so that $V_i$ is disjoint from the Lagrangian cocore $L_j$ in (1.21) when $i \neq j$, and $V_i$ intersects $L_i$ transversely at a unique point. Since $M_{p,q,r}$ is the smooth fiber of a Lefschetz fibration on $\mathbb{C}^4$ obtained by suspending

$\tilde{t}_{p,q,r} : \mathbb{C}^3 \to \mathbb{C}$ once instead of twice, it is in general not clear whether $\mathcal{V}_{p,q,r}$ is a trivial extension of the directed $A_\infty$-algebra $\mathcal{A}_{p,q,r}$, see Lemma 3.1.1 [2]. However, combining Theorem 1.2.1 above with Theorem 4 of [36], we obtain the following formality result of $A_\infty$-algebras:

**Corollary 1.2.1.** *There is a quasi-isomorphism*

$$\mathcal{V}_{p,q,r} \cong \mathcal{A}_{p,q,r} \oplus \mathcal{A}^{\vee}_{p,q,r}[-3]. \tag{1.25}$$

*In particular, the Fukaya $A_\infty$-algebra $\mathcal{V}_{p,q,r}$ is formal.*

The detailed proof of the above corollary is given in Section 3.3. Note that when $r = 2$, $M_{p,q,r}$ can be regarded as the smooth fiber of the *double suspension* of the Lefschetz fibration on $\mathbb{C}^2$ defined by the Morsification of

$$(x^{p-2} - y^2)(x^2 - \lambda y^{q-2}), \tag{1.26}$$

see Section 2.2.5 of [60]. In this case, the formality of $\mathcal{V}_{p,q,2}$ follows from a result of Seidel [92]. We will give a brief sketch of his argument in Section 3.1, and the fact that $M_{p,q,r}$ being a double suspension is necessary for his formality theorem is also explained in detail there.

As a consequence of Theorem 1.2.1 and Corollary 1.2.1, we have quasi-isomorphisms

$$R\mathrm{Hom}_{\mathcal{V}_{p,q,r}}(\Bbbk, \Bbbk) \cong CE^*(\Lambda_{p,q,r}), R\mathrm{Hom}_{CE^*(\Lambda_{p,q,r})}(\Bbbk, \Bbbk) \cong \mathcal{V}_{p,q,r} \tag{1.27}$$

between $\mathbb{Z}$-graded $A_\infty$-algebras, which, as we have mentioned at the end of Section 1.1, follows from a version Koszul duality between the Calabi-Yau completion and its corresponding cyclic completion, see Section 2.5 for a detailed explanation. In fact, it can be proved that the Fukaya category $\mathcal{F}(M_{p,q,r})$ is generated by vanishing cycles, see Corollary 1.2.2.

[2]The author is grateful to Ailsa Keating for pointing out this.

**Remark 1.2.2.** *It is worthy to compare the approach that we adopt in this thesis to understand the wrapped Fukaya category with the dictionary of Seidel in Section 6 of [93], which describes the wrapped Fukaya category $\mathcal{W}(M)$ of the total space of a Lefschetz fibration $\pi : M \to \mathbb{C}$ as the result of a categorical localization. More precisely, one can form from the $A_\infty$-algebras $\mathcal{A}_\pi$ and $\mathcal{V}_F$ a curved $A_\infty$-algebra*

$$\mathcal{D}_M := \mathcal{A}_\pi \oplus t\mathcal{V}_F[[t]] \tag{1.28}$$

*with curvature $\mu^0_{\mathcal{D}} = t \cdot \mathrm{id}$, $\mathrm{id} \in \Bbbk \subset \mathcal{D}^0_M$, where $F$ is the smooth fiber of $\pi$, and $t$ is a formal variable of degree 2. It is conjectured by Seidel in [93] and proved by Ganatra-Maydanskiy in the appendix of [16] that we have a quasi-isomorphism*

$$\mathcal{W}_M \cong T(\overline{\mathcal{D}}_M[1])^\vee, \tag{1.29}$$

*which expresses the linear dual of the wrapped Fukaya $A_\infty$-algebra as a tensor coalgebra, where $\overline{\mathcal{D}}_M \cong \mathcal{D}_M/\Bbbk$. Therefore in order to show that $\mathcal{W}_M$ is the $A_\infty$-Koszul dual of $\mathcal{V}_M$, it suffices to identify $\mathcal{D}_M$ with the $A_\infty$-algebra $\mathcal{V}_M$ with vanishing curvature. The simplest case when $M$ is a 4-dimensional $A_n$ Milnor fiber is studied by Pomerleano in Section 9 of [84].*

As an application of the Koszul duality functor introduced in [13], we have the following split-generation result of the compact Fukaya categories of $M_{p,q,r}$.

**Corollary 1.2.2.** *The Fukaya category $\mathcal{F}(M_{p,q,r})$ is split-generated by the vanishing cycles $V_1, \cdots, V_{p+q+r-1}$.*

This will be proved in Section 3.4. Note that when $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, the Milnor fiber $M_{p,q,r}$ can be defined by a quasi-homogeneous polynomial with weights $p, q, r, 2$. Since

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{2} = \frac{3}{2} \neq 1, \tag{1.30}$$

the split-generation of $\mathcal{F}(M_{p,q,r})$ by vanishing cycles follows from a theorem of Seidel (cf. [90, 91]).

For earlier results concerning the generation of the compact Fukaya categories of plumbings, see [8]. In general, not much is known about the compact Fukaya category of a Liouville manifold, despite the fact that significant effort has been devoted to understand the wrapped Fukaya category of a Weinstein manifold [16, 46, 47].

Our last application of Theorem 1.2.1 ties Koszul duality between Fukaya $A_\infty$-algebras up with categorical dynamics in the sense of Seidel [90], which is another major topic of this thesis. Recall that for any Liouville manifold $M$ with $c_1(M) = 0$, we can define, using a Hamiltonian function which is quadratic on the cylindrical end, the symplectic cohomology $SH^*(M)$, which carries the structure of a BV (Batalin-Vilkovisky) algebra. According to Seidel-Solomon [103], a *quasi-dilation* is a pair $(b, h) \in SH^1(M) \times SH^0(M)^\times$ such that under the action of the BV operator $\Delta : SH^*(M) \to SH^{*-1}(M)$, it satisfies

$$\Delta(hb) = h. \tag{1.31}$$

When $h = 1$, the class $b \in SH^1(M)$ is called a *dilation*. A proof of the following fact is given in Section 3.5.

**Corollary 1.2.3.** *The Liouville manifold $M_{p,q,r}$ admits a quasi-dilation over any field $\mathbb{K}$.*

As we have remarked above, $M_{p,q,2}$ is Liouville isomorphic to an affine conic bundle over $\mathbb{C}^2$, therefore the existence of a quasi-dilation follows from an iterative application of the Lefschetz fibration argument due to Seidel-Solomon, see Lecture 19 of [90].

## 1.3 Exact Calabi-Yau structures

For the remaining part of this chapter, assume that $\mathrm{char}(\mathbb{K}) = 0$. The $n$-Calabi-Yau completion $\Pi_n(\mathcal{A})$ mentioned above is an example of an *exact Calabi-Yau* algebra. In view of Theorem 1.2.1, it is natural to ask the following:

**Question 1.3.1.** *Let $M$ be a Liouville manifold. When is the wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_M$ exact Calabi-Yau?*

To address the above question, we start by recalling the basic notions. Let $\mathcal{A}$ be an $A_\infty$-algebra over some semisimple ring $\Bbbk$, which is homologically smooth. It can be regarded as a bimodule over itself, which is known as the diagonal bimodule and by slight abuse of notation, we will still denote it by $\mathcal{A}$. By our assumption, $\mathcal{A}$ is a perfect bimodule. Its dual bimodule, $\mathcal{A}^\vee$, is defined as

$$\mathcal{A}^\vee := R\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e), \tag{1.32}$$

where $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{op}$. Recall that $\mathcal{A}$ is a *weak smooth $n$-Calabi-Yau algebra* if there exists a non-degenerate Hochschild cycle $\eta \in CH_{-n}(\mathcal{A})$, i.e. a cocycle which induces an isomorphism

$$\mathcal{A}^\vee[n] \cong \mathcal{A} \tag{1.33}$$

between $\mathcal{A}$-bimodules.

Associated to $\mathcal{A}$ there is a long exact sequence [76]

$$\cdots \to HC_{-*-1}(\mathcal{A}) \xrightarrow{S} HC_{-*+1}(\mathcal{A}) \xrightarrow{B} HH_{-*}(\mathcal{A}) \xrightarrow{I} HC_{-*}(\mathcal{A}) \to \cdots \tag{1.34}$$

relating Hochschild and cyclic homologies of $\mathcal{A}$, which is known as *Connes' long exact sequence* [76].

**Definition 1.3.1** (Keller)**.** *A weak smooth $n$-Calabi-Yau structure on $\mathcal{A}$ is said to be exact if the Hochschild homology class $[\eta]$ lies in the image of Connes' map $B : HC_{-n+1}(\mathcal{A}) \to HH_{-n}(\mathcal{A})$.*

Notice that the notion of an exact Calabi-Yau structure is strictly more restrictive than a *smooth Calabi-Yau structure* in the sense of Kontsevich-Vlassopoulos [29, 69], which is defined as a negative cyclic cycle $\tilde{\eta} \in CC^{-}_{-n}(\mathcal{A})$ whose induced Hochschild cycle in $CH_{-n}(\mathcal{A})$ under the inclusion map of homotopy fixed points $\iota : CC^{-}_{*}(\mathcal{A}) \to CH_{*}(\mathcal{A})$ defines a weak smooth $n$-Calabi-Yau structure on $\mathcal{A}$. This can be easily seen from the following commutative diagram [76]:

$$\begin{tikzcd} HC_{-n+1}(\mathcal{A}) \arrow[r] \arrow[dr, "B"'] & HC^{-}_{-n}(\mathcal{A}) \arrow[d, "{[\iota]}"] \\ & HH_{-n}(\mathcal{A}) \end{tikzcd} \tag{1.35}$$

Just as a Calabi-Yau structure $[\eta] \in HH_{-n}(\mathcal{A})$ is the noncommutative analogue of a holomorphic volume form $\Omega$, the existence of a lift $[\tilde{\eta}]$ in $HC^{-}_{-n}(\mathcal{A})$ corresponds to the (trivial) fact that $\Omega$ is necessarily closed. Since Connes' differential $B$ is the noncommutative analogue of the de Rham differential, the exact Calabi-Yau condition imposed on $\mathcal{A}$ is analogous to the exactness of $\Omega$ as a differential form. This explains the terminology.

We remark that an important class of examples of exact Calabi-Yau $A_\infty$-algebras is the so called *superpotential algebras* introduced by Ginzburg [50], which is roughly a dg algebra whose underlying associative algebra is modelled on some localization of the path algebra, and whose differential is specified by a superpotential lying in the commutator quotient $\mathcal{A}/[\mathcal{A}, \mathcal{A}]$, see Section 2.2 for details. As a special case, we have the Ginzburg dg algebra $\mathcal{G}(Q, w)$ associated to a quiver with potential $(Q, w)$, see

[50]. In particular, it follows from Theorem 1.2.1 that the wrapped Fukaya categories $\mathcal{W}(M_{p,q,r})$ are exact Calabi-Yau.

However, it is in general not true that the wrapped Fukaya category of any Weinstein manifold carries an exact Calabi-Yau structure. The first set of such counterexamples is found by Davison [30], who studied the fundamental group algebra $\mathbb{K}[\pi_1(Q)]$ of a $K(\pi,1)$ space $Q$, and showed that when $Q$ is a hyperbolic manifold, $\mathbb{K}[\pi_1(Q)]$ is not exact Calabi-Yau. Note that for a closed manifold which is topologically $K(\pi,1)$, we have a quasi-isomorphism

$$\mathcal{W}_{T^*Q} := CW^*(T_q^*Q, T_q^*Q) \cong \mathbb{K}[\pi_1(Q)] \tag{1.36}$$

between (formal) $A_\infty$-algebras [4].

There is a closed string counterpart of the notion of an exact Calabi-Yau structure, which makes use of an $S^1$-equivariant version of the symplectic cohomology $SH^*(M)$. Denote by $SH^*_{S^1}(M)$ the $S^1$-*equivariant symplectic cohomology* so that $SH^*_{S^1}(pt) \cong \mathbb{K}((u))/u\mathbb{K}[[u]]$, see Section 7.1 for a detailed account of its construction. Analogous to Connes' long exact sequence (1.34), the symplectic cohomologies $SH^*(M)$ and $SH^*_{S^1}(M)$ fit into the following Gysin type long exact sequence [19]:

$$\cdots \to SH^{*-1}(M) \xrightarrow{\mathbf{I}} SH^{*-1}_{S^1}(M) \xrightarrow{\mathbf{S}} SH^{*+1}_{S^1}(M) \xrightarrow{\mathbf{B}} SH^*(M) \to \cdots \tag{1.37}$$

where the composition $\mathbf{B} \circ \mathbf{I}$ gives the BV operator $\Delta$.

**Remark 1.3.1.** *Note that in the above, we have used the bold letters* $\mathbf{I}$, $\mathbf{B}$ *and* $\mathbf{S}$ *to denote the corresponding maps $I$, $B$ and $S$ in Connes' long exact sequence (1.34), in order to emphasize that we are dealing with closed string invariants. As a convention, we will use the notations* $\mathbb{I}$, $\mathbb{B}$ *and* $\mathbb{S}$ *for the open string counterparts of the maps* $\mathbf{I}$,

**B** *and* **S**. *In particular, there is a long exact sequence*

$$\cdots \to HC_{-*-1}(\mathcal{W}(M)) \xrightarrow{\mathbb{S}} HC_{-*+1}(\mathcal{W}(M)) \xrightarrow{\mathbb{B}} HH_{-*}(\mathcal{W}(M)) \xrightarrow{\mathbb{I}} HC_{-*}(\mathcal{W}(M)) \to \cdots, \tag{1.38}$$

*which is simply (1.34) applied to the wrapped Fukaya category* $\mathcal{W}(M)$.

To relate the two long exact sequences (1.37) and (1.38), we implement the *cyclic open-closed string map*

$$[\widetilde{OC}] : HC_*(\mathcal{W}(M)) \to SH^{*+n}_{S^1}(M) \tag{1.39}$$

constructed by Ganatra [43], from which one obtains the following geometric interpretation of an exact Calabi-Yau structure on $\mathcal{W}(M)$:

**Proposition 1.3.1.** *Let $M$ be a non-degenerate Liouville manifold, its wrapped Fukaya category $\mathcal{W}(M)$ carries an exact Calabi-Yau structure if and only if the connecting map* **B** : $SH^1_{S^1}(M) \to SH^0(M)$ *in (1.37) hits an invertible element* $h \in SH^0(M)^\times$.

In the above, the non-degeneracy condition on a Liouville manifold is introduced by Ganatra in [44], which ensures that the ordinary open-closed map

$$[OC] : HH_*(\mathcal{W}(M)) \to SH^{*+n}(M) \tag{1.40}$$

is an isomorphism. A Liouville manifold $M$ is said to be *non-degenerate* if there is a finite collection of Lagrangians $\{L_i\}$ in $M$ such that $OC$ restricted to the full $A_\infty$-subcategory $\mathcal{L}(M) \subset \mathcal{W}(M)$ generated by $\{L_i\}$ hits the identity $1 \in SH^0(M)$. Combining the generation of the wrapped Fukaya category by Lagrangian cocores proved in [23, 47] and Abouzaid's generation criterion [2], any Weinstein manifold is non-degenerate since one can take $\mathcal{L}(M)$ to be the full $A_\infty$-subcategory of cocores.

A class $\tilde{b} \in SH^1_{S^1}(M)$ satisfying

$$\mathbf{B}(\tilde{b}) = h \in SH^0(M)^\times \tag{1.41}$$

will be called a *cyclic dilation* (cf. Definition 7.2.1). Now Proposition 1.3.1 says that an exact Calabi-Yau structure on $\mathcal{W}(M)$ of a non-degenerate Liouville manifold $M$ induces a cyclic dilation. Note that if $M$ admits a quasi-dilation in the sense of (1.31), then it also admits a cyclic dilation $\tilde{b}$ which arises as the image of $hb$ under the map $\mathbf{I} : SH^1(M) \to SH^1_{S^1}(M)$. It is then natural to ask whether the converse is true. We postpone the discussions about whether the cyclic dilation condition (1.41) is strictly weaker than the quasi-dilation condition (1.31) to Section 1.4, and look here an immediate geometric implication by assuming the existence of a cyclic dilation.

Let $L \subset M$ be a closed exact Lagrangian submanifold, equipped with a rank 1 local system $\nu$ so that the isomorphism $SH^*(T^*L) \cong H^*(\mathcal{L}L;\nu)$ holds [3], where $\mathcal{L}L$ denotes the free loop space of $L$. There is an $S^1$-equivariant version of Viterbo functoriality, namely the ($S^1$-equivariant lift of) the *Cieliebak-Latschev map* constructed by Cohen-Ganatra [29]

$$[\widetilde{CL}] : SH^*_{S^1}(M) \to H^{S^1}_{n-*}(\mathcal{L}L;\nu), \tag{1.42}$$

see Section 8.1. Combined with Proposition 1.3.1, we can reinterpret Davison's non-existence result mentioned in Section 1.3 in the following slightly more general form:

**Proposition 1.3.2.** *Let $M$ be a Liouville manifold which admits a cyclic dilation, then it does not contain any closed exact Lagrangian submanifold $L \subset M$ which is hyperbolic.*

Analogous to what Seidel and Solomon have done in the case of dilations and quasi-dilations [103], one can use Lefschetz fibrations to produce more examples of Liouville manifolds which admit cyclic dilations starting from the known ones. More precisely, we prove in Section 9.2 the following:

**Theorem 1.3.1.** *Let $M$ be a $2n$-dimensional Liouville manifold, such that $n > 1$ is odd. Suppose that $\pi : M \to \mathbb{C}$ is an exact symplectic Lefschetz fibration with smooth fiber $F$. If $F$ admits a cyclic dilation, then the same is true for the total space $M$.*

## 1.4 Trichotomy of affine varieties

The well-known trichotomy of Riemannian manifolds says that positively curved, flat, and negatively curved manifolds have distinct geometric behaviours. In symplectic topology, there is an analogy of this trichotomy for Liouville manifolds. Geometrically, this can be understood by studying the existence and abundance of $J$-holomorphic maps $u : S \to \overline{M}$ with finite energy in the interior of the associated Liouville domain $\overline{M}$, where $S$ is a punctured sphere, see [79].

For simplicity, we restrict our attention to the case when $M$ is an $n$-dimensional smooth affine variety over $\mathbb{C}$, equipped with the restriction of the constant symplectic form on the ambient affine space, in which case the aforementioned trichotomy has a numerical description in terms of the log Kodaira dimension $\kappa(M)$ mentioned in Section 1.1. We shall be particularly interested in three cases, namely when $\kappa(M) = -\infty$, $\kappa(M) = 0$ (in which case $M$ is known as *log Calabi-Yau*); and $\kappa(M) = n$ (in which case $M$ is called *log general type*). These should be thought of as analogues of positively curved, flat and negatively curved Riemannian manifolds, respectively. The existence question of a cyclic dilation (or equivalently, an exact Calabi-Yau structure on $\mathcal{W}(M)$) will be considered separately in these three cases.

First, let $M$ be a smooth affine variety with $\kappa(M) = -\infty$. An important class of such manifolds is given by the Milnor fibers $M_{a_1,\cdots,a_{n+1}} \subset \mathbb{C}^{n+1}$ associated to the Brieskorn singularities

$$z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} = 0, \tag{1.43}$$

where $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$. In this thesis, we will study the simplest non-trivial case, namely a Fermat affine cubic 3-fold $M_{3,3,3,3} \subset \mathbb{C}^4$.

**Theorem 1.4.1.** *The manifold $M_{3,3,3,3}$ admits a cyclic dilation.*

This will be proved in Section 2.5 using essentially algebraic arguments. Abstractly, one should think of Theorem 1.4.1 as a consequence of the Koszul duality between the compact and the wrapped Fukaya categories of $M_{3,3,3,3}$. In fact, we will prove along the way that the Fukaya $A_\infty$-algebra of a basis of vanishing cycles $\mathcal{F}_{M_{3,3,3,3}}$ is Koszul dual to the Fukaya $A_\infty$-algebra of the corresponding Lagrangian cocores $\mathcal{W}_{M_{3,3,3,3}}$, see Proposition 9.1.5.

Another key point of the proof is to show that up to quasi-isomorphism, the wrapped Fukaya $A_\infty$-algebra of $M_{3,3,3,3}$ is concentrated in non-positive degrees, which is expected to be true for any $M_{a_1,\cdots,a_{n+1}}$ with $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$, although the verification is more involved in general.

Note that if $a_i \geqslant 3$ for all $i$, then $M_{a_1,\cdots,a_{n+1}}$ does not admit a quasi-dilation. This is argued in Example 2.7 of [96] for dilations, and the argument there extends trivially to the more general case of quasi-dilations (a sketch is given in the proof of Corollary 9.1.2). In particular, Theorem 1.4.1 shows that the existence of a cyclic dilation is strictly weaker than having a quasi-dilation.

**Remark 1.4.1.** *More interesting examples of Liouville manifolds admitting cyclic dilations are established in the recent work of Zhou [119]. In particular, his result* *implies that $M_{a,\cdots,a}$ admits a cyclic dilation as long as $n \geqslant a$, therefore generalizes Theorem 1.4.1 above. Our method has the advantage that it is applicable to examples beyond complements of smooth divisors in projective varieties, see Proposition 9.1.2.*

Second, we consider the case when $M$ is a smooth log Calabi-Yau variety. These manifolds provide important local examples for testing the validity of mirror symmetry and have been studied extensively in the existing literature [6, 48, 49, 53, 83]. As an illustration to the general situation, we consider here the simplest case when $\dim_{\mathbb{C}}(M) = 2$, and make the following observation:[3]

**Proposition 1.4.1.** *Let $M$ be an affine log Calabi-Yau surface with maximal boundary, then $M$ admits a cyclic dilation if and only if it admits a quasi-dilation.*

*Proof.* With our assumptions, one can arrange so that the Conley-Zehnder indices of the periodic orbits are 0, 1 and 2, see for example [49, 83]. In particular, the cochain complex $SC^*(M)$ defining the symplectic cohomology $SH^*(M)$ is supported in these three degrees. Thus any cyclic dilation can only come from a cocycle in $SC^1(M)$, see our discussions in Section 7.2 for details. □

We expect the same to be true in higher dimensions, although no insights can be drawn from the argument above.

Finally, let us take a look at the case when $M$ is a smooth affine variety of log general type. To get some concrete examples, one can take any Milnor fiber $M_{a_1,\cdots,a_{n+1}}$ as above, but now with $\sum_{i=1}^{n+1}\frac{1}{a_i} < 1$. In complex dimension 2, the Milnor fibers associated to Arnold's 14 exceptional unimodal singularities are affine surfaces

[3]The author thanks Daniel Pomerleano for suggesting this approach to prove Proposition 1.4.1, which greatly simplifies the original argument.

of log general type, since they are complements of ample divisors in K3 surfaces, see [72].

Via the Abel-Jacobi map, we can embed a genus two curve $\Sigma_2$ in its Jacobian variety $J(\Sigma_2)$, let $M$ be the complement in $J(\Sigma_2)$ of the image of $\Sigma_2$. Clearly, $M$ is log general type. On the other hand, since there is an embedding $D^*T^2\#D^*T^2 \hookrightarrow M$ from the plumbing of two copies of the disc cotangent bundles over $T^2$ into $M$ as a Liouville subdomain, there is a genus two exact Lagrangian surface in $M$. One can therefore use Proposition 1.3.2 to conclude that there is no cyclic dilation in $SH^1_{S^1}(M)$. This example can be generalized to the case when $M$ is the complement of a nearly tropical hypersurface in the abelian variety $(\mathbb{C}^*)^n/\Gamma$, where $\Gamma \subset \mathbb{R}^n$ is a lattice, see Section 10 of [6].

It seems that genus two exact Lagrangian surfaces can also be established in the 4-dimensional Milnor fibers $M_{a_1,a_2,a_3}$ with $\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}<1$, by imitating the strategy of Keating in [60]. However, it is not true that hyperbolic exact Lagrangian submanifolds can always be constructed in varieties of log general type. For instance, this is the case of the complement $M$ of $n+2$ generic hyperplanes in $\mathbb{CP}^n$, with $n \geqslant 2$. These manifolds are known as higher dimensional pair-of-pants, and are studied extensively in the context of mirror symmetry, see for example [70]. Since $M$ is uniruled by $(n+2)$-punctured holomorphic spheres, similar argument as in the proof of Theorem 1.7.5 of [39] excludes the existence of hyperbolic Lagrangians in $M$.

We expect that a smooth affine variety of log general type can never admit a cyclic dilation, and will prove the following general statement in Section 9.3:

**Theorem 1.4.2.** *Let $M$ be a smooth affine variety of log general type which contains an exact Lagrangian $K(\pi,1)$, then it does not admit a cyclic dilation.*

Theorem 1.4.2 is not helpful in general as there are many contractible affine varieties of log general type: one of them is the Ramanujam surface studied in [102].

## 1.5 Categorical dynamics and disjoint Lagrangian spheres

**Warning** The results displayed in this section are still work in progress, and the details of the proofs will not be included as part of this thesis.

To see how the notion of a cyclic dilation can be useful in the study of symplectic topology, we start with a brief overview of the theory of categorical dynamics [91, 93, 95, 96, 103]. Since we are not interested in proving the general set up of the theory, various assumptions will be imposed here to keep the situation as simple as possible.

Given a quiver with potential $(Q, w)$, one can associate to it two $A_\infty$-algebras over $\Bbbk := \bigoplus_{v \in Q_0} \mathbb{K} e_v$. One of them is the (completed) Ginzburg dg algebra $\widehat{\mathcal{G}}(Q, w)$, while the other one, denoted as $\mathcal{B}(Q, w)$, is introduced by Kontsevich-Soibelman [67]. $\mathcal{B}(Q, w)$ is related to $\widehat{\mathcal{G}}(Q, w)$ via Koszul duality. More precisely, there are quasi-isomorphisms

$$\mathcal{B}(Q, w) \cong R\mathrm{Hom}_{\mathcal{G}(Q,w)}(\Bbbk, \Bbbk), \widehat{\mathcal{G}}(Q, w) \cong R\mathrm{Hom}_{\mathcal{B}(Q,w)}(\Bbbk, \Bbbk) \tag{1.44}$$

as $\mathbb{Z}$-graded $A_\infty$-algebras over $\Bbbk$, where $\Bbbk := \bigoplus_{i \in Q_0} \mathbb{K} e_i$ is the semisimple ring consisting of copies of $\mathbb{K}$ indexed by the set of vertices $Q_0$ of $Q$. Koszul duality between $\mathcal{B}(Q, w)$ and $\mathcal{G}(Q, w)$ induces an isomorphism between their Hochschild cohomologies:

$$HH^*(\mathcal{B}(Q, w)) \cong HH^*(\mathcal{G}(Q, w)). \tag{1.45}$$

Since $\mathcal{B}(Q,w)$ is by definition a cyclic $A_\infty$-algebra, and $\mathcal{G}(Q,w)$ is exact Calabi-Yau, their Hochschild cohomologies carry naturally induced BV structures, and (1.45) is a BV algebra isomorphism.

Let us assume temporarily that the superpotential $w$ is homogeneous and consists only of cubic terms, which in particular implies the formality of the $A_\infty$-structure on $\mathcal{B}(Q,w)$. Note that this is exactly the situation for the quiver with potential $(Q_{p,q,r}, w_{p,q,r})$ mentioned in Remark 1.2.1. We will further assume that the ground field $\mathbb{K} = \mathbb{C}$. The graded associative algebra $\mathcal{B}(Q,w)$ then carries a rational $\mathbb{C}^*$-action, which has weight $i$ on the degree $i$ part. According to [95], this $\mathbb{C}^*$-action enables us to define a bigraded refinement $\mathcal{B}(Q,w)^{\mathbf{perf}}$ of the $A_\infty$-category $\mathcal{B}(Q,w)^{perf}$ of perfect $A_\infty$-modules over $\mathcal{B}(Q,w)$. The $\mathbb{C}^*$-action on $\mathcal{B}(Q,w)$ induces at the infinitesimal level a Hochschild cocycle $eu_{\mathcal{B}} \in CH^1(\mathcal{B}(Q,w))$. This particular noncommutative vector field is known as the *Euler vector field*. Under the BV operator, $eu_{\mathcal{B}}$ goes to a non-zero scalar multiple of the identity. Any object $\mathcal{E}$ of $\mathcal{B}(Q,w)^{perf}$ which is *rigid* and *simple*, i.e. which satisfies

$$H^0(\hom_{\mathcal{B}(Q,w)^{perf}}(\mathcal{E},\mathcal{E})) \cong \mathbb{C}, H^1(\hom_{\mathcal{B}(Q,w)^{perf}}(\mathcal{E},\mathcal{E})) \cong 0, \tag{1.46}$$

is $\mathbb{C}^*$-equivariant, and therefore defines an object in the category $\mathcal{B}(Q,w)^{\mathbf{perf}}$. In particular, it is infinitesimally equivariant with respect to $eu_{\mathcal{B}}$. For any two such objects $\mathcal{E}_0$ and $\mathcal{E}_1$, $eu_{\mathcal{B}}$ defines an endomorphism of $H^*(\hom_{\mathcal{B}(Q,w)^{perf}}(\mathcal{E}_0,\mathcal{E}_1))$, from which one recovers the weight grading on $\mathcal{B}(Q,w)^{\mathbf{perf}}$.

Geometrically, let $M$ be a $2n$-dimensional Weinstein manifold, and assume that its wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_M$ is quasi-isomorphic to the Ginzburg dg algebra $\mathcal{G}(Q,w)$. In particular, $M$ admits a cyclic dilation. To simplify our discussions here, assume further that the Fukaya categories $\mathcal{F}(M)$ and $\mathcal{W}(M)$ are Koszul dual, so that the endomorphism algebra $\mathcal{F}_M$ of a set of split-generators in $\mathcal{F}(M)$ can be identified with $\mathcal{B}(Q, w)$. Our assumptions therefore ensure that the closed-open string map

$$[CO] : SH^*(M) \to HH^*(\mathcal{F}(M)) \tag{1.47}$$

is an isomorphism.

Via the quasi-isomorphism $\mathcal{F}_M \cong \mathcal{B}(Q, w)$ and the inverse of (1.47), the Euler vector field $eu_{\mathcal{B}}$ gives rise to a quasi-dilation $b \in SH^1(M)$. In other words, the infinitesimal symmetry $b$ integrates to a *dilating* $\mathbb{C}^*$-action on the Fukaya category $\mathcal{F}(M)$ in the sense of [91]. For any two $\mathbb{C}^*$-equivariant objects $L_0, L_1$ of $\mathcal{F}(M)$, this enables us to define an endomorphism of their Floer cohomology algebra $HF^*(L_0, L_1)$, which equips $HF^*(L_0, L_1)$ with an additional $\mathbb{C}$-grading by generalized eigenspaces.

More generally, one can start directly with a quasi-dilation $b \in SH^1(M)$, and consider the infinitesimal deformation of the objects in $\mathcal{F}(M)$ along $b$. For $b$-equivariant Lagrangian submanifolds $L_0, L_1 \subset M$, the infinitesimal action of $b$ still defines a derivation $\Phi_{\tilde{L}_0,\tilde{L}_1}$ on the $\mathbb{C}$-algebra $HF^*(L_0, L_1)$, which allows one to define the *$q$-intersection number*

$$L_0 \bullet_q L_1 := \mathrm{Str}\left(e^{\log(q)\Phi_{\tilde{L}_0,\tilde{L}_1}}\right). \tag{1.48}$$

Setting $q = 1$ one recovers the ordinary topological intersection number $[L_0] \cdot [L_1]$.

The geometric significance of the $q$-intersection number becomes obvious if we take $L_0 = L_1$ to be an odd-dimensional Lagrangian sphere, in which case $[L] \cdot [L] = 0$ while

$$L \bullet_q L = 1 - q \tag{1.49}$$

does not vanish. Thus one may expect that $q$-intersection numbers can be used to detect the non-trivialness of the homology classes of odd-dimensional Lagrangian spheres. More generally, we have the following special case of Eliashberg's *Regular Lagrangian Conjecture* (cf. Section 5 of [38]):

**Conjecture 1.5.1** (folklore)**.** *Let $M$ be any Weinstein manifold, and $L \subset M$ a closed exact Lagrangian submanifold with vanishing Maslov class. Then its homology class $[L] \in H_n(M;\mathbb{Z})$ is primitive.*

It is proved by Seidel in [96] that Conjecture 1.5.1 holds for Milnor fibers of isolated hypersurface singularities $\{p(z_1, \cdots, z_{n+1}) = 0\}$ in $\mathbb{C}^{n+1}$ satisfying

$$\operatorname{rank}\left(D^2 p_{z=0}\right) \geqslant 4. \tag{1.50}$$

These Milnor fibers have nice properties in the sense that they admit dilations over any field $\mathbb{K}$, and the geometry of the dilations are simple enough in the sense that if we define the symplectic cohomology $SH^*(M)$ as the direct limit of the Floer cohomology $HF^*(\lambda)$ of the Hamiltonian functions $H_\lambda : M \to \mathbb{R}$ of slope $\lambda$ on the cylindrical end, then the dilation $b \in HF^1(\lambda)$ appears at the smallest possible slope $\lambda > 0$.

It is natural to ask whether there is a similar theory after removing the assumption that $\mathcal{B}(Q,w)$ (and thus $\mathcal{F}_M$) is formal. Note that as a consequence of non-formality, the aforementioned $\mathbb{C}^*$-action on $\mathcal{B}(Q,w)$ does not preserve the $A_\infty$-structure.

In general, given any Liouville manifold $M$ with a cyclic dilation $\tilde{b} \in SH^1_{S^1}(M)$, one can try to imitate Seidel-Solomon's construction by making use of the higher order closed-open string maps introduced by Ganatra [43]. Due to the existence of certain obstruction terms (corresponding to additional boundary strata in the relevant moduli spaces), it is in general not possible to obtain endomorphisms on Floer cohomology groups $HF^*(L_0, L_1)$ of simply-connected Lagrangian submanifolds.

However, for a single Lagrangian sphere $L \subset M$ with dimension $n \geqslant 3$, the obstructions vanish and there is a well-defined endomorphism $\Phi_{\tilde{L},\tilde{L}}$ on $HF^*(L, L)$ such that $\mathrm{Str}\left(\Phi_{\tilde{L},\tilde{L}}\right) \neq 0$. This fact enables us to prove the following:

**Theorem 1.5.1** (work in progress)**.** *Let $M$ be a $2n$-dimensional Weinstein manifold, where $n \geqslant 3$ is odd, and $c_1(M) = 0$. Assume that $M$ admits a cyclic dilation $\tilde{b} \in SH^1_{S^1}(M)$, and satisfies an additional property $(\widetilde{H})$. Suppose that $L_1, \cdots, L_r \subset M$ is a collection of pairwise disjoint Lagrangian spheres, then their homology classes $[L_1], \cdots, [L_r] \in H_n(M; \mathbb{K})$ span a subspace of dimension $\geqslant r/2$. In particular,*

$$r \leqslant 2 \dim H^n(M; \mathbb{K}), \tag{1.51}$$

*and for any Lagrangian sphere $L \subset M$, $[L] \neq 0$.*

Here, property $(\widetilde{\mathrm{H}})$ imposes additional restrictions on the cohomology class $\tilde{b}$. More precisely, there is a continuation map $HF^1_{S^1}(\lambda) \to SH^1_{S^1}(M)$ from the $S^1$-equivariant Floer cohomology of a Hamiltonian with slope $\lambda$ on the cylindrical end $[r_0, \infty) \times \partial\overline{M}$ for $r_0 \gg 0$ to the $S^1$-equivariant symplectic cohomology, see (7.33). Roughly speaking, property $(\widetilde{\mathrm{H}})$ says that the class $\tilde{b}$ appears already in $HF^1_{S^1}(\lambda)$, with $\lambda > 0$ being relatively small with respect to the minimal period of the Reeb orbits on the contact boundary $\partial\overline{M}$.

This property is expected to be true for many Weinstein manifolds with cyclic dilations, including the Milnor fibers $M_{a_1,\cdots,a_{n+1}}$ with $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$. For example, it follows from the argument in Section 5.2 of [119] that it holds for the Milnor fibers $M_{a,\cdots,a}$ with $n \geqslant a$. In fact, these manifolds also satisfy that $\mathbf{B}(\tilde{b}) = 1$ for the cyclic dilation $\tilde{b}$, in which case we have the following stronger result:

**Corollary 1.5.1** (work in progress)**.** *Let $M$ and $L_1, \cdots, L_r$ be as in Theorem 1.5.1, and assume further that the cyclic dilation $\tilde{b}$ satisfies $h = 1$ in (1.41). If $\mathbb{K} \subset \mathbb{R}$, then the homology classes $[L_i] \in H_n(M; \mathbb{K})$ are linearly independent.*

Theorem 1.5.1 and Corollary 1.5.1 generalize the results of Seidel (Theorems 1.6 and 1.7 of [96]) since the existence of a cyclic dilation, although restrictive, is a much weaker assumption than requiring that $M$ admits a dilation.

**Example 1.5.1.** *For the Milnor fiber $M_{3,3,3,3}$, Theorem 1.5.1 applies and yields an upper bound of 32 for the number of pairwise disjoint Lagrangian spheres. Since $M_{3,3,3,3}$ admits a cyclic dilation with $h = 1$, it follows from Corollary 1.5.1 that this bound can be improved to 16. Taking further into account of the fact that the subspace of $H_3(M_{3,3,3,3}; \mathbb{Q}) \cong \mathbb{Q}^{16}$ spanned by the classes $[L_1], \cdots, [L_r]$ of disjoint Lagrangian spheres is isotropic with respect to the intersection form yields the bound $r \leqslant 11$. On the other hand, in view of the intersection pattern of a basis of vanishing cycles in $M_{3,3,3,3}$ depicted in Figure 9.1, one finds that the Lagrangian spheres*

$$V_{11}, V_{12}, V_{13}, V_{21}, V_{22}, V_{23}, V_{31}, V_{32}, V_{33}, V_{\gamma\gamma} \tag{1.52}$$

*are disjoint. It would be interesting to determine whether $r \leqslant 10$ or not.*

# Chapter 2

# Algebraic Preliminaries

## 2.1 Calabi-Yau algebra and quiver with potential

Let $\mathbb{K}$ be any field. As mentioned in Section 1.5, given a quiver with potential $(Q, w)$, one can associate to it two different (but related by Koszul duality) $\mathbb{Z}$-graded $A_\infty$-algebras over $\Bbbk$: a homologically smooth $A_\infty$-algebra $\mathcal{G}(Q, w)$, and a proper $A_\infty$-algebra $\mathcal{B}(Q, w)$. The first one is a smooth $n$-Calabi-Yau algebra, while the second one is a proper $n$-Calabi-Yau algebra.

Let $\mathcal{B}$ be a proper $A_\infty$-algebra over $\Bbbk$. A *proper $n$-Calabi-Yau structure* on $\mathcal{B}$ is defined as a degree $n$ chain map

$$\widetilde{tr} : CC_*(\mathcal{B}) \to \mathbb{K}[-n] \tag{2.1}$$

whose projection to the Hochschild complex, $tr : CH_*(\mathcal{B}) \to \mathbb{K}[-n]$, defines a *weak proper $n$-Calabi-Yau structure*, i.e. it induces a perfect pairing

$$\begin{aligned} &H^*\left(\hom_{\mathcal{B}^{perf}}(\mathcal{P}, \mathcal{Q})\right) \otimes H^{n-*}\left(\hom_{\mathcal{B}^{perf}}(\mathcal{Q}, \mathcal{P})\right) \xrightarrow{[\mu^2_{\mathcal{B}}]} H^n\left(\hom_{\mathcal{B}^{perf}}(\mathcal{Q}, \mathcal{Q})\right) \\ &\to HH_n(\mathcal{B}) \xrightarrow{[tr]} \mathbb{K}. \end{aligned} \tag{2.2}$$

The notion of a proper Calabi-Yau structure is closely related to that of a *cyclic $A_\infty$-structure*, which is by definition a chain level perfect pairing

$$\langle \cdot, \cdot \rangle_{\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathbb{K}[-n] \tag{2.3}$$

such that the induced correlation functions $\langle \mu_{\mathcal{B}}^k(\cdot, \cdots, \cdot), \cdot \rangle$ are strictly (graded) cyclically symmetric for each $k \geqslant 1$. When $\mathrm{char}(\mathbb{K}) = 0$, these two notions are essentially equivalent in the sense that any cyclic $A_\infty$-algebra $\mathcal{B}$ over $\mathbb{k}$ has a canonically defined proper Calabi-Yau structure, and any proper Calabi-Yau structure on an $A_\infty$-algebra $\mathcal{B}$ determines a quasi-isomorphism between $\mathcal{B}$ and a cyclic $A_\infty$-algebra, the latter fact is due to Kontsevich-Soibelman [68].

For simplicity, we only recall the definitions of $\mathcal{G}(Q, w)$ and $\mathcal{B}(Q, w)$ in the case when $n = 3$, the general case is similar and details can be found in Section 9.3 of [111]. Alternatively, one can view it as a special case of the construction given in Section 2.2 below. $\mathcal{G}(Q, w)$ is in fact a dg algebra, which is originally introduced by Ginzburg in [50]. Let $Q = (Q_0, Q_1)$ be a finite quiver with the set of vertices $Q_0$ and the set of arrows $Q_1$. By definition, a *potential* on $Q$ is an element

$$w \in \mathbb{K}Q/[\mathbb{K}Q, \mathbb{K}Q] \tag{2.4}$$

in the space of all cyclic paths in $Q$. Without loss of generality, we can always assume $w$ to be *reduced*, namely there is no summand in $w$ which is a cycle in $Q$ with length $< 3$. The *Ginzburg algebra* $\mathcal{G}(Q, w)$ associated to $(Q, w)$ is the dg algebra freely generated over $\mathbb{K}$ by

- the arrows $a \in Q_1$ with $\deg(a) = 0$;
- the opposite arrows $a^*$ of $a$ with $\deg(a^*) = -1$;

- loops $z_v$ for every vertex $v \in Q_0$, with $\deg(z_v) = -2$.

The differential $d$ on $\mathcal{G}(Q,w)$ is defined to be

$$da = 0, da^* = \frac{{}^\circ\partial \tilde{w}}{\partial a}, d(\sum_{v \in Q_0} z_v) = \sum_{a \in Q_1} [a, a^*] \tag{2.5}$$

on the set of generators, where $\tilde{w}$ is the sum of all cyclic permutations of $w$, and ${}^\circ\partial$ is the circular derivative introduced by Kontsevich, which is by definition

$$\frac{{}^\circ\partial \tilde{w}}{\partial a} = \sum_{\tilde{w} = uav} vu. \tag{2.6}$$

One can then extend $d$ to a differential on the whole algebra $\mathcal{G}(Q,w)$ by graded Leibniz rule. Note that $\mathcal{G}(Q,w)$ can be regarded as a dg algebra over $\Bbbk = \bigoplus_{v \in Q_0} \mathbb{K} e_v$, with its $\Bbbk$-bimodule structure induced from the path algebra $\mathbb{K}\widetilde{Q}$ of the extended quiver $\widetilde{Q}$, which is obtained by adjoining opposite arrows $a^*$ and loops $z_v$ to the original quiver $Q$.

Start with the same quiver with potential $(Q,w)$ as above. As a graded $\Bbbk$-module, the $\mathbb{Z}$-graded $A_\infty$-algebra $\mathcal{B}(Q,w)$ is given by

$$\mathcal{B}(Q,w) := \bigoplus_{i,j \in Q_0} \mathbb{K}^{\delta_{ij}} \oplus V_{ij}^\vee[-1] \oplus V_{ji}^\vee[-2] \oplus \mathbb{K}^{\delta_{ij}}[-3], \tag{2.7}$$

where $V_{ij}$ is the (trivially graded) vector space with basis consisting of the arrows in $Q_1$ which start from the vertex $i$ and end at the vertex $j$, and $V_{ji}$ is the vector space generated by the opposite of the arrows in $V_{ij}$. The $A_\infty$-operations

$$\mu_{\mathcal{B}}^k : \mathcal{B}(Q,w)^{\otimes k} \to \mathcal{B}(Q,w)[2-k] \tag{2.8}$$

are defined explicitly by

$$\mu_{\mathcal{B}}^k(g_k^\vee, \dots, g_1^\vee) = (-1)^{(k-1)|g_k^\vee| + \cdots + 2|g_3^\vee| + |g_2^\vee|} \sum_g \mathrm{Coeff}_{g_k \cdots g_1}(dg) \cdot g^\vee, \tag{2.9}$$

where the sum on the right-hand side above is taken over all the generators of the Ginzburg algebra $\mathcal{G}(Q,w)$, and $\mathrm{Coeff}_{g_k\cdots g_1}(dg)$ is the coefficient of $g_k\dots g_1$ in the differential of $g$ in $\mathcal{G}(Q,w)$, which is determined by the potential $w$. Moreover, the grading of $g^\vee$ is given by

$$|g^\vee| = 1 - \deg(g) \tag{2.10}$$

in the $A_\infty$-algebra $\mathcal{B}(Q,w)$.

Consider the pairing $(\cdot,\cdot)$ on the vector space generated by the degree 0 and degree $-1$ generators of $\mathcal{G}(Q,w)$, which satisfies

- $(g_i,g_j) = -(-1)^{|g_1||g_2|}(g_j,g_i)$;
- $(g_i,g_j) = 0$ unless $t(g_i) = h(g_j)$ and $t(g_j) = h(g_i)$, where $h(g)$ and $t(g)$ are respectively the head and tail of an arrow or opposite arrow $g$;
- the matrix with entries given by $(g_i,g_j)$ is invertible.

Let $\langle\cdot,\cdot\rangle$ be the dual pairing of $(\cdot,\cdot)$, in this way we get a non-degenerate pairing

$$\langle\cdot,\cdot\rangle_{\mathcal{B}} : \mathcal{B}(Q,w)\times\mathcal{B}(Q,w)\to\mathbb{K}[-3] \tag{2.11}$$

defined by

$$\langle g_1^\vee, g_2^\vee\rangle_{\mathcal{B}} = (-1)^{|g_1^\vee|}\langle g_1^\vee, g_2^\vee\rangle. \tag{2.12}$$

Using the definition of $\mu_{\mathcal{B}}^k$, one can verify that

$$\left\langle \mu_{\mathcal{B}}^k(g_k^\vee,\dots,g_1^\vee), g_0^\vee\right\rangle_{\mathcal{B}} = (-1)^{k+|g_k^\vee|(|g_{k-1}^\vee|+\cdots+|g_0^\vee|)}\left\langle \mu_{\mathcal{B}}^k(g_{k-1}^\vee,\dots,g_0^\vee), g_k^\vee\right\rangle_{\mathcal{B}}, \tag{2.13}$$

which shows that $\mathcal{B}(Q,w)$ is a cyclic $A_\infty$-algebra with respect to the pairing $\langle\cdot,\cdot\rangle_{\mathcal{B}}$.

## 2.2 Superpotential algebras

In this section, we discuss a generalization of the Ginzburg dg algebra $\mathcal{G}(Q,w)$ associated to the quiver with potential $(Q,w)$ defined in Section 2.1, which is called a superpotential algebra. As we have mentioned in Section 1.3, superpotential algebras form an important class of exact Calabi-Yau algebras. For simplicity, we assume $\mathrm{char}(\mathbb{K}) = 0$ in this section.

Let $\mathcal{A}$ be a finitely generated unital dg algebra over $\Bbbk$, denote by $\Omega^1_{\mathcal{A}}$ the $\mathcal{A}$-bimodule of differential 1-forms on $\mathcal{A}$. $\mathcal{A}$ is said to be *quasi-free* if $\Omega^1_{\mathcal{A}}$ is projective. For example, any path algebra of a quiver is quasi-free, so is its localization. Assume from now on that $\mathcal{A}$ is quasi-free, define the space of de Rham differential forms

$$DR_{\Bbbk}(\mathcal{A}) := T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})/[T_{\mathcal{A}}(\Omega^1_{\mathcal{A}}), T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})], \tag{2.14}$$

where $T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})$ is the tensor algebra of $\Omega^1_{\mathcal{A}}$ over $\mathcal{A}$, so in particular $DR_{\Bbbk}(\mathcal{A})$ carries the structure of a dg algebra. Denote by $Der_{\Bbbk}(\mathcal{A},\mathcal{A})$ the dg vector space of $\Bbbk$-linear (super)derivations on $\mathcal{A}$. For a closed 2-form $\omega \in DR^2_{\Bbbk}(\mathcal{A})$, we have a map

$$i_\omega : Der_{\Bbbk}(\mathcal{A},\mathcal{A}) \to DR^1_{\Bbbk}(\mathcal{A}) \tag{2.15}$$

defined by contracting $\omega$ with every derivation in $Der_{\Bbbk}(\mathcal{A},\mathcal{A})$. We say that $\omega$ is *symplectic* if $i_\omega$ is an isomorphism. Let $\mathbb{D}er_{\Bbbk}(\mathcal{A}) := Der_{\Bbbk}(\mathcal{A},\mathcal{A}\otimes\mathcal{A})$ be the bimodule of *double derivations* of $\mathcal{A}$, analogous to (2.15) we have a map

$$\iota_\omega : \mathbb{D}er_{\Bbbk}(\mathcal{A}) \to \Omega^1_{\mathcal{A}}. \tag{2.16}$$

A symplectic form $\omega$ is *bisymplectic* if $\iota_\omega$ is an isomorphism.

Given any $a \in \mathcal{A}$ and let $\omega \in DR^2_{\Bbbk}(\mathcal{A})$ be bisymplectic, consider the double derivation $H_a \in \mathbb{D}er(\mathcal{A})$ defined by $\iota_{H_a}\omega = Da$. Using $H_a$ one can define a bracket

$\{\cdot,\cdot\}$ on $\mathcal{A}$ by

$$\{a_1, a_2\} = \circ(H_{a_1}(a_2)), \tag{2.17}$$

where $\circ : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication on $\mathcal{A}$. It can be checked that $\{\cdot,\cdot\}$ descends to a Lie bracket on $\mathcal{A}/[\mathcal{A},\mathcal{A}]$, and it defines an action of $\mathcal{A}/[\mathcal{A},\mathcal{A}]$ on $\mathcal{A}$ by derivations.

On the other hand, given $\theta \in Der_{\Bbbk}(\mathcal{A},\mathcal{A})$, we can define the Lie derivative $L_\theta$ on $T_{\mathcal{A}}(\Omega^1_{\mathcal{A}})$ by

$$L_\theta(a) = \theta(a), L_\theta(Da) = D(\theta(a)) \tag{2.18}$$

for any $a \in \mathcal{A}$ and $Da \in \Omega^1_{\mathcal{A}}$. Clearly, $L_\theta$ descends to a map on the quotient $DR_{\Bbbk}(\mathcal{A})$.

Consider the triple $(\mathcal{A}, \omega, \theta)$, where $\mathcal{A} = T_{\mathcal{B}}(\mathcal{M})$ is connected and non-positively graded, with $\mathcal{B}$ being a quasi-free associative algebra concentrated in degree 0, and $\mathcal{M}$ a $\mathcal{B}$-module. The 2-form $\omega \in DR^2_{\Bbbk}(\mathcal{A})$ is a bisymplectic form which is homogeneous with respect to the grading induced from $\mathcal{A}$, and $\theta \in Der_{\Bbbk}(\mathcal{A},\mathcal{A})$ has cohomological degree 1, which satisfies $L_\theta\omega = 0$ and $\theta^2 = 0$. One can associate to this data a dg algebra $\mathcal{G}(\omega,\theta)$, called *Ginzburg dg algebra*, whose underlying graded algebra is given by the free product $\mathcal{A} * \Bbbk[t]$, where $|t| = |\omega| - 1$. The differential $d$ on $\mathcal{G}(\omega,\theta)$ is defined by considering the non-commutative moment map

$$\mu_{nc} : DR^2_{\Bbbk}(\mathcal{A})_{cl} \to \overline{\mathcal{A}} := \mathcal{A}/\Bbbk \tag{2.19}$$

on the space of closed cyclic 2-forms $DR^2_{\Bbbk}(\mathcal{A})_{cl} \subset DR^2_{\Bbbk}(\mathcal{A})$, which satisfies $D(\mu_{nc}(\omega)) = \iota_\Delta\omega$, where $\Delta \in \mathbb{D}er(\mathcal{A})$ is the double derivation

$$\Delta(a) = a \otimes 1 - 1 \otimes a. \tag{2.20}$$

Under the assumption that $\mathcal{A}$ is connected, $\mu_{nc}$ lifts to a map

$$\tilde{\mu}_{nc} : DR^2_{\Bbbk}(\mathcal{A})_{cl} \to [\mathcal{A},\mathcal{A}]. \tag{2.21}$$

With the above notations, we have

$$da = \theta(a), dt = \tilde{\mu}_{nc}(\omega). \tag{2.22}$$

Note that our assumptions on the derivation $\theta$ ensures that $d^2 = 0$, so the dg algebra $\mathcal{G}(\omega, \theta)$ is well-defined.

**Definition 2.2.1.** *The Ginzburg dg algebra $\mathcal{G}(\omega, \theta)$ defined above is a superpotential algebra if $\theta = \{w, \cdot\}$, where $w \in \mathcal{A}/[\mathcal{A}, \mathcal{A}]$ is called the superpotential.*

If one takes $\mathcal{A}$ above to be the path algebra $\mathbb{K}\widetilde{Q}$ of the double $\widetilde{Q}$ of some quiver $Q = (Q_0, Q_1)$ obtained by adding a reverse $a^*$ to all the arrows $a \in Q_1$ (if $a$ is a cycle of odd degree, then $a^* = a$), the above construction recovers the Ginzburg dg algebra (or the *dg preprojective algebra*, in the terminology of [111]) $\mathcal{G}(Q, w)$ associated to the quiver with potential $(Q, w)$ defined in Section 2.1.

However, there are also many superpotential algebras which are not of the form $\mathcal{G}(Q, w)$, a typical example is the fundamental group algebra $\mathbb{K}[\pi_1(T^3)]$, see Example 4.3.4 of [30]. Another counterexample is the following:

**Example 2.2.1.** *Consider the associative algebra $\mathbb{K}[x, y][(xy-1)^{-1}]$, regarded as a trivially graded dg algebra with vanishing differential. It is known that this algebra has a superpotential description. In fact, consider the path algebra $\mathbb{K}\widetilde{Q}_{1,0,0}$ of the double of the quiver $Q_{1,0,0}$ which consists of a single vertex and a unique cycle $x$, see Figure 2.1. Let*

$$\mathcal{A} = \mathbb{K}\langle x, y\rangle[(xy-1)^{-1}] \tag{2.23}$$

*be the localization of $\mathbb{K}\widetilde{Q}_{1,0,0}$ at $xy-1$, where the generator $y$ represents the reverse of the cycle $x$ in the quiver $\widetilde{Q}_{1,0,0}$. Since $\mathbb{K}\widetilde{Q}_{1,0,0}$ is quasi-free, so is $\mathcal{A}$. In this way, we have identified $\mathbb{K}[x, y][(xy-1)^{-1}]$ with the dg algebra $(\mathcal{A} * \mathbb{K}[t], d)$, so that $dx = dy = 0$ and $dt = xy - yx$. The superpotential vanishes for dimension reasons.*

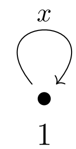


Figure 2.1: The quiver $Q_{1,0,0}$

**Theorem 2.2.1** (Theorem 4.3.8 of [30]). *Let $\mathcal{G}(\omega, \theta)$ be a Ginzburg dg algebra so that $\omega$ has cohomological degree $-n+2$, then $\mathcal{G}(\omega, \theta)$ has a smooth $n$-Calabi-Yau structure which is exact.*

## 2.3 Cyclic completions

For the remaining part of this chapter, $\mathbb{K}$ will be an arbitrary field. We recall the construction of the cyclic completion of an $A_\infty$-algebra from [88]. In certain cases, this construction yields the $A_\infty$-algebra $\mathcal{B}(Q, w)$ defined in Section 2.1. This is nothing else but the chain level refinement of trivial extensions of (graded) associative algebras.

Let $\Bbbk$ be a semisimple ring. Suppose that $\mathcal{A}$ is a $\mathbb{Z}$-graded $A_\infty$-algebra over $\Bbbk$, with $\mu^k_{\mathcal{A}} : \mathcal{A}^{\otimes k} \to \mathcal{A}[2-k]$ being its structure maps, then one can associate to $\mathcal{A}$ another $A_\infty$-algebra $\mathcal{B}$ which as an $\mathcal{A}$-bimodule is given by

$$\mathcal{A} \oplus \mathcal{A}^\vee[-n], \tag{2.24}$$

where $\mathcal{A}^\vee[-n]$ is the dual of the diagonal $\mathcal{A}$-bimodule, whose bimodule operations will be denoted by

$$\mu^{s|1|r}_{\mathcal{A}^\vee[-n]} : \mathcal{A}^{\otimes s} \otimes \mathcal{A}^\vee[-n] \otimes \mathcal{A}^{\otimes r} \to \mathcal{A}^\vee[1-r-s-n]. \tag{2.25}$$

The $A_\infty$-operations

$$\mu^k_{\mathcal{B}} : \mathcal{B}^{\otimes k} \to \mathcal{B}[2-k] \tag{2.26}$$

on the trivial extension are then defined to be the direct sum of $\mu^k_{\mathcal{A}}$ with $\mu^{i-1|1|k-i}_{\mathcal{A}^\vee[-n]}$. More precisely,

$$\begin{aligned}
&\mu^k_{\mathcal{B}}\left((a_k, a_k^\vee), \ldots, (a_1, a_1^\vee)\right) := \Big(\mu^k_{\mathcal{A}}(a_k, \ldots, a_1), \\
&\sum_{i=1}^{k} (-1)^{|a_1|+\cdots+|a_{i-1}|-i+2} \mu^{i-1|1|k-i}_{\mathcal{A}^\vee[-n]}(a_k, \ldots, a_{i+1}, a_i^\vee, a_{i-1}, \ldots, a_1)\Big),
\end{aligned} \tag{2.27}$$

where $a_j \in \mathcal{A}$ and $a_j^\vee \in \mathcal{A}^\vee[-n]$.

The $A_\infty$-algebra $\mathcal{B}$ defined above is called the *n-cyclic completion* of $\mathcal{A}$. In many cases, the $A_\infty$ structure $\{\mu^\bullet_{\mathcal{B}}\}$ on $\mathcal{B}$ defined above is cyclic and extends the (usually) non-cyclic $A_\infty$-structure $\{\mu^\bullet_{\mathcal{A}}\}$ on $\mathcal{A}$. For example, it follows from Theorem 4.2 of [88] that this is the case when $\mathcal{A}$ is the endomorphism algebra of any object in the derived category of coherent sheaves on a smooth proper scheme. As another example, consider the following situation (cf. [62], Section 6.9):

Let $\mathcal{A} \cong \mathbb{K}\overrightarrow{Q}/\overrightarrow{I}$ be a finite-dimensional graded associative algebra over $\mathbb{K}$ defined by a quiver $\overrightarrow{Q} = (\overrightarrow{Q}_0, \overrightarrow{Q}_1)$ with the set of vertices $\overrightarrow{Q}_0$ and the set of arrows $\overrightarrow{Q}_1$, together with a 2-sided ideal $\overrightarrow{I} \subset \mathbb{K}\langle \overrightarrow{Q}_1 \rangle^{\otimes 2}$ generated by quadratic relations (so in particular the $A_\infty$-structure on $\mathcal{A}$ is formal). In this case, the 3-cyclic completion $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}^\vee[-3]$ has a simple realization. More precisely, denote by $\{\rho_1, \ldots, \rho_r\}$ the set of relations which generate the ideal $\overrightarrow{I}$, and by $y_j$ an arrow which reverses the compositions of the arrows appeared in $\rho_j$. Then $\mathcal{B}$ is the cyclic $A_\infty$-algebra defined by the quiver with potential $(Q, w)$ (cf. Section 2.1), where $Q_0 = \overrightarrow{Q}_0$,

$$Q_1 = \overrightarrow{Q}_1 \cup \{y_1, \ldots, y_r\} \tag{2.28}$$

and

$$w = \sum_{1 \leqslant j \leqslant r} y_j \rho_j. \tag{2.29}$$

**Proposition 2.3.1.** *Let $\mathcal{A} \cong \mathbb{K}\overrightarrow{Q}/\overrightarrow{I}$ be as above, then the $A_\infty$-structure on its 3-cyclic completion $\mathcal{B}$ is formal.*

*Proof.* This is obvious since $\mu^k_{\mathcal{B}}$ is defined additively in terms of $\mu^k_{\mathcal{A}}$, but $\mathcal{A}$ is formal as an $A_\infty$-algebra. □

## 2.4 Calabi-Yau completions

We recall here another algebraic construction associated to $A_\infty$-algebras due to Keller [62], which serves as a generalization to the Ginzburg dg algebra $\mathcal{G}(Q, w)$. This is Koszul dual to Segal's cyclic completion.

Let $\mathcal{A}$ be a $\mathbb{Z}$-graded $A_\infty$-algebra over the semisimple ring $\mathbb{k} = \bigoplus_{i=1}^r \mathbb{K}e_i$. Without loss of generality we can assume that $\mu^k_{\mathcal{A}} = 0$ for $k \geqslant 3$, namely it is differential graded, since in general we can always find a dg algebra $\mathcal{A}'$ which is quasi-isomorphic to $\mathcal{A}$ and apply the construction described below to $\mathcal{A}'$. Its *$n$-Calabi-Yau completion* is defined to be the tensor dg algebra

$$\Pi_n(\mathcal{A}) := T_{\mathcal{A}}(\mathrm{Res}(\mathcal{A})^\vee[n-1]), \tag{2.30}$$

where the $\mathcal{A}$-bimodule $\mathrm{Res}(\mathcal{A})^\vee$ is called the *inverse dualizing complex.*

Let us illustrate the above definition under the assumption that $\mathcal{A}$ is a *semi-free dg algebra* over the field $\mathbb{K}$, which means its underlying graded algebra is freely generated over $\mathbb{K}$, with the set of generators given by $\{a_1, \ldots, a_m\}$. Consider $\Omega^1_{\mathcal{A}}$, the bimodule of differentials on $\mathcal{A}$, which has $Da_1, \ldots, Da_m$ as its basis, with $D : \mathcal{A} \to \Omega^1_{\mathcal{A}}$

being the universal derivation. Recall that we have a short exact sequence

$$0 \to \Omega^1_{\mathcal{A}} \xrightarrow{\alpha} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\circ} \mathcal{A} \to 0, \tag{2.31}$$

where $\alpha(Da) = a \otimes 1 - 1 \otimes a$, and $\circ$ is the multiplication on $\mathcal{A}$. In this case, $\mathrm{Res}(\mathcal{A})$ is the mapping cone of $\alpha$, which gives a semi-free bimodule resolution of $\mathcal{A}$ with basis $\{e_1 \otimes e_1, \cdots, e_r \otimes e_r, Da_1[1], \cdots, Da_m[1]\}$. $\mathrm{Res}(\mathcal{A})^\vee$ is the dual bimodule of $\mathrm{Res}(\mathcal{A})$.

If $\phi : \mathcal{A} \to \mathcal{A}'$ is a quasi-isomorphism between dg algebras over $\Bbbk$, then there is an induced quasi-isomorphism $\Pi_n(\phi) : \Pi_n(\mathcal{A}) \to \Pi_n(\mathcal{A}')$.

**Theorem 2.4.1** (Keller [62, 63])**.** *If $\mathcal{A}$ is homologically smooth, then its $n$-Calabi-Yau completion $\Pi_n(\mathcal{A})$ is an exact $n$-Calabi-Yau algebra.*

We now specialize the above construction to the case of quiver algebras. Denote again by $\overrightarrow{Q} = (\overrightarrow{Q}_0, \overrightarrow{Q}_1)$ a finite quiver. Let $\mathcal{A} = \mathbb{K}\overrightarrow{Q}/\overrightarrow{I}$ be the graded associative algebra defined by a quadratic ideal $\overrightarrow{I} \subset \mathbb{K}\langle\overrightarrow{Q}_1\rangle^{\otimes 2}$. Here we assume in addition that

*$\mathcal{A}$ has global dimension less than or equal to 2.*

From these data we can construct a quiver with potential $(Q, w)$ exactly as in Section 2.3. Namely one starts with the quiver $\overrightarrow{Q}$ and adds arrows to it according to the relations which generate $\overrightarrow{I}$.

In this case we have the following result due to Keller, which gives a concrete realization of the 3-Calabi-Yau completion $\Pi_3(\mathcal{A})$ as the Ginzburg algebra associated to $(Q, w)$.

**Proposition 2.4.1** (Theorem 6.10 of [62])**.** *Under the above assumptions, the 3-Calabi-Yau completion $\Pi_3(\mathcal{A})$ is quasi-isomorphic to the Ginzburg dg algebra $\mathcal{G}(Q, w)$.*

## 2.5 Koszul duality of quiver algebras

We explain the Koszul duality between the compact and smooth 3-Calabi-Yau algebras $\mathcal{B}(Q,w)$ and $\mathcal{G}(Q,w)$ defined in Section 2.1 associated to the same quiver with potential $(Q,w)$, where

*$w$ is homogeneous and contains only cubic terms.*

We refer the interested readers to Section 2 of [36] for a detailed introduction to the algebraic preliminaries concerning Koszul duality.

In this section, we will follow the convention of [77], and work with $\mathbb{Z}\times\mathbb{Z}$-graded augmented $A_\infty$-algebras $(\mathcal{A},\varepsilon)$ over a semisimple ring $\Bbbk$, where $\varepsilon:\mathcal{A}\to\Bbbk$ is an augmentation. This means that all the $A_\infty$-operations $\{\mu_{\mathcal{A}}^k\}_{k\geqslant 1}$ have bidegree $(2-k,0)$. Let $\phi:\mathcal{A}\to\mathcal{B}$ be a morphism between bigraded $A_\infty$-algebras, which consists of a sequence of $\Bbbk$-linear maps $\phi_k:\mathcal{A}^{\otimes k}\to\mathcal{B}$, then $\phi_k$ has bidegree $(1-k,0)$. The second grading on $\mathcal{A}$ will be referred to as the *Adams grading*, it is preserved by all the $A_\infty$-operations $\{\mu_{\mathcal{A}}^k\}_{k\geqslant 1}$. With respect to the bigrading, as a $\Bbbk$-bimodule $\mathcal{A}$ decomposes as $\mathcal{A}=\bigoplus_{i,j}\mathcal{A}^{i,j}$. We say that $\mathcal{A}$ is *locally finite* if each $\mathcal{A}^{i,j}$ is finite-dimensional over $\Bbbk$.

As a $\Bbbk$-bimodule, the (bigraded) Koszul dual $E(\mathcal{A})$ of $\mathcal{A}$ is defined explicitly by

$$E(\mathcal{A})^{m,n}:=\bigoplus_{d\geqslant 1}\bigoplus_{\sum_{k=1}^d i_k=m,\sum_{k=1}^d j_k=n}(\overline{\mathcal{A}}[1]^{\#})^{i_1,j_1}\otimes\cdots\otimes(\overline{\mathcal{A}}[1]^{\#})^{i_d,j_d}, \tag{2.32}$$

where $\overline{\mathcal{A}}:=\ker(\varepsilon)$ is the augmentation ideal, $\#$ denotes the graded linear dual, and the shift functor $[1]$ acts on the first grading. Note that $E(\mathcal{A})$ carries te structure of an augmented dg algebra, with its differential induced from the $A_\infty$-structure on $\mathcal{A}$.

Note that a map $\phi:\mathcal{A}\to\mathcal{B}$ between $\mathbb{Z}\times\mathbb{Z}$-graded augmented $A_\infty$-algebras is a quasi-isomorphism if and only if the induced coaugmented dg coalgebra map

$\mathrm{B}\phi : \mathrm{B}\mathcal{A} \to \mathrm{B}\mathcal{B}$ on their bar constructions is a quasi-isomorphism. This can be proved using the spectral sequence associated to the *word length filtration* on the complex $\mathrm{B}\mathcal{A}$, see Section 2.2.1 of [36]. Recall that by definition, $E(\mathcal{A}) = (\mathrm{B}\mathcal{A})^{\#}$, therefore a quasi-isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between augmented $A_\infty$-algebras induces a quasi-isomorphism $E(\mathcal{A}) \to E(\mathcal{B})$.

**Theorem 2.5.1** ([77], Theorem 2.4)**.** *Let $\mathcal{A}$ be a locally finite augmented $A_\infty$-algebra. If its Koszul dual $E(\mathcal{A})$ is also locally finite, then $E(E(\mathcal{A}))$ is quasi-isomorphic to $\mathcal{A}$.*

**Remark 2.5.1.** *In [77], there is no locally finiteness assumption on $\mathcal{A}$. However, this assumption is essential for the argument presented in [77], see for example Lemma 9 and Theorem 17 of [36], where such an assumption is included.*

**Proposition 2.5.1.** *Let $\mathcal{B}(Q,w)$ and $\mathcal{G}(Q,w)$ be the compact and smooth Calabi-Yau 3-algebras defined by the same quiver with potential $(Q,w)$. We can equip them with Adams gradings so that there are quasi-isomorphisms between bigraded $A_\infty$-algebras*

$$E(\mathcal{B}(Q,w)) \cong \mathcal{G}(Q,w), E(\mathcal{G}(Q,w)) \cong \mathcal{B}(Q,w). \tag{2.33}$$

*Proof.* To begin with, we equip $\mathcal{B}(Q,w)$ with a trivial bigrading $(0,j)$, where the first grading is always fixed to be 0, and the Adams grading is the total grading on the $\mathbb{Z}$-graded $A_\infty$-algebra $\mathcal{B}(Q,w)$.

Note that although the original grading on $E(\mathcal{B}(Q,w)) = \mathcal{G}(Q,w)$ is usually not locally finite, we can equip $\mathcal{G}(Q,w)$ with a bigrading by declaring the following:

- the original arrows $a$ in $Q_1$ have bigrading $(1,-1)$;
- the opposite arrows $a^*$ to $a$ have bigrading $(1,-2)$;
- the loops $z_v$ at the vertex $v$ have bigrading $(1,-3)$,

so that the total grading recovers the original grading on $\mathcal{G}(Q,w)$. By our assumption that every term in the potential $w$ is cubic, we see that the differential $d$ on $\mathcal{G}(Q,w)$ has bidegree $(1,0)$ with respect to the bigrading defined above.

Note that $(\mathcal{B}(Q,w),\varepsilon_{\mathcal{B}})$ is a bigraded augmented $A_\infty$-algebra with $\varepsilon_{\mathcal{B}}:\mathcal{B}(Q,w)\to\Bbbk$ being the projection to the degree 0 part. By (2.32) we have

$$E(\mathcal{B}(Q,w)) = T\left(\overline{\mathcal{B}(Q,w)}[1]^\vee\right). \tag{2.34}$$

It follows from our definitions in Section 2.1 that the right-hand side above is quasi-isomorphic to the Ginzburg dg algebra $\mathcal{G}(Q,w)$ equipped with the bigrading specified above. This proves the first quasi-isomorphism.

Since $\mathcal{G}(Q,w)$ is locally finite with respect to the double grading specified above, and $\mathcal{B}(Q,w)$ is clearly locally finite, we can apply Theorem 2.5.1 to $\mathcal{B}(Q,w)$, which gives

$$\mathcal{B}(Q,w)\cong E(E(\mathcal{B}(Q,w)))\cong E(\mathcal{G}(Q,w)). \tag{2.35}$$

$\square$

**Remark 2.5.2.** *The bigrading used in the above proof has potential applications in showing the primitivity of the homology classes of Lagrangian homology spheres in $M_{p,q,r}$ when $\mathbb{K}=\mathbb{C}$. In fact, Corollaries 1.2.1 and 1.2.2 boil the question of classifying Lagrangian homology spheres in $M_{p,q,r}$ down to that of classifying $\mathbb{C}^*$-equivariant $A_\infty$-modules over a quiver algebra $\mathcal{B}_{p,q,r}$, see Section 3.3. Meanwhile, the dg category of perfect $A_\infty$-modules over $\mathcal{B}_{p,q,r}$ admits a bigraded refinement, which comes essentially from the bigrading on $\mathcal{B}_{p,q,r}$ specified in the proof of the above Proposition. One can then imitate the argument of* [*98*].

**Proposition 2.5.2.** *Assume in addition that $H^*(\mathcal{G}(Q,w))$ is finite dimensional in* *each fixed degree, we have quasi-isomorphisms*

$$\mathcal{B}(Q,w)^! \cong \mathcal{G}(Q,w), \mathcal{G}(Q,w)^! \cong \mathcal{B}(Q,w) \tag{2.36}$$

*as $\mathbb{Z}$-graded $A_\infty$-algebras.*

*Proof.* According to Section 2.3 of [115], in general we have

$$\mathcal{B}(Q,w)^! \cong \widehat{\mathcal{G}}(Q,w), \tag{2.37}$$

where $\widehat{\mathcal{G}}(Q,w)$ is the *completed Ginzburg algebra* associated to the quiver with potential $(Q,w)$, namely the Ginzburg dg algebra $\mathcal{G}(Q,w)$ completed with respect to the path length in $\mathbb{K}\widetilde{Q}$, with $\widetilde{Q}$ being the "double" of $Q$ obtained by adding to $Q$ the reversed arrows $a^*$ for each $a \in Q_1$ and the loops $z_v$ for each $v \in Q_0$.

To conclude the proof, we need to show that with our assumptions, $\widehat{\mathcal{G}}(Q,w)$ is quasi-isomorphic to the uncompleted Ginzburg algebra $\mathcal{G}(Q,w)$. This can be seen by considering the filtration $F^\bullet H^*(\mathcal{G}(Q,w))$ on cohomology induced by the path length filtration on $\mathcal{G}(Q,w)$. By our standing assumption that the potential $w$ consists only of cubic terms, we see that the differentials of the generators in $\mathcal{G}(Q,w)$ consist of homogeneous terms with respect to the path length filtration, thus the filtration $F^\bullet$ on $H^*(\mathcal{G}(Q,w))$ is Hausdorff, which means that the completion map $\mathcal{G}(Q,w) \to \widehat{\mathcal{G}}(Q,w)$ is cohomologically injective. On the other hand, since $H^*(\mathcal{G}(Q,w))$ is finite dimensional in each degree, we conclude that the filtration $F^\bullet$ on $H^*(\mathcal{G}(Q,w))$ is complete, therefore $\mathcal{G}(Q,w) \to \widehat{\mathcal{G}}(Q,w)$ is cohomologically surjective. See Section 5.4 of [114]. □

## 2.6 Complexes with $S^1$-actions

The remaining two sections of this chapter are a little bit distant from the previous five sections, and they are included here to serve as preliminary materials for Chapters

7 to 9. This section defines the notion of an $S^1$-complex, and the exposition here follows closely Section 2 of [43]. Let $C_{-*}(S^1)$ be the dg algebra of chains on $S^1$, there is a quasi-equivalence of dg algebras $C_{-*}(S^1) \cong \mathbb{K}[t]/(t^2), |t| = -1$. The following definition is introduced in [19] and [43].

**Definition 2.6.1** ([43], Definitions 1 and 2)**.** *An $S^1$-complex, or a chain complex with an $A_\infty$ $S^1$-action, is a strictly unital $A_\infty$-module $\mathcal{P}$ over $C_{-*}(S^1)$. Equivalently, it is a graded $\mathbb{K}$-vector space equipped with operations $\delta_j^{\mathcal{P}} : \mathcal{P} \to \mathcal{P}[1-2j]$, with $\delta_0^{\mathcal{P}} := d^{\mathcal{P}}$ being the differential on $\mathcal{P}$, such that for each $k \geqslant 0$, the equation*

$$\sum_{j=0}^{k} \delta_j^{\mathcal{P}} \delta_{k-j}^{\mathcal{P}} = 0 \tag{2.38}$$

*holds. If the $A_\infty$-module $\mathcal{P}$ is a dg module, then it is called a strict $S^1$-complex.*

We will abbreviate $\delta_j^{\mathcal{P}}$ to $\delta_j$ as long as there is no confusion.

$S^1$-complexes form a dg category $C_{-*}(S^1)^{umod}$, whose morphisms we will now recall. Let $A := C_{-*}(S^1)$ be the quadratic algebra with a degree $-1$ generator, denote by $\varepsilon : A \to \mathbb{K}$ the trivial augmentation, and by $\overline{A} := \ker(\varepsilon)$ the augmentation ideal. Let $\mathcal{P}$ and $\mathcal{Q}$ be strictly unital $A_\infty$-modules over $A$. A *unital pre-morphism* of degree $d$ from $\mathcal{P}$ to $\mathcal{Q}$ is a collection of maps

$$F^{d|1} : \overline{A}^{\otimes d} \otimes \mathcal{P} \to \mathcal{Q}[k-d] \tag{2.39}$$

for each $d \geqslant 0$. Or equivalently, it can be expressed as a set of maps $\{F^d\}$, with

$$F^d := F^{d|1}(t, \cdots, t, \cdot) : \mathcal{P} \to \mathcal{Q}[k-2d]. \tag{2.40}$$

The space of pre-morphisms in each degree form the graded vector space of morphisms between the objects $\mathcal{P}$ and $\mathcal{Q}$ in the dg category $A^{umod}$, which will be denoted by $R\mathrm{Hom}_{S^1}(\mathcal{P}, \mathcal{Q})$. Since $A^{umod}$ is a dg category, there is a differential $\partial$ on

$R\mathrm{Hom}_{S^1}(\mathcal{P}, \mathcal{Q})$, which is defined by

$$(\partial F)^s := \sum_{i=0}^{s} F^i \circ \delta_{s-i}^{\mathcal{P}} - (-1)^{\deg(F)} \sum_{j=0}^{s} \delta_{s-j}^{\mathcal{Q}} \circ F^j. \tag{2.41}$$

An $S^1$-*complex homomorphism* is a pre-morphism which is closed under $\partial$. A homomorphism $F : \mathcal{P} \to \mathcal{Q}$ between $S^1$-complexes is a *quasi-isomorphism* if the induced map $[F^0] : H^*(\mathcal{P}) \to H^{*+\deg(F)}(\mathcal{Q})$ on cohomologies is an isomorphism.

Let $\mathcal{P}$ and $\mathcal{Q}$ be $S^1$-complexes, one can define their (derived) tensor product, which is another $S^1$-complex $\mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}$. To do this, note that we can view $\mathcal{Q}$ as a right $A_\infty$-module over $A$ since $A$ is commutative. The chain complex $\mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}$ has underlying vector space

$$\bigoplus_{d \geqslant 0} \mathcal{Q} \otimes \overline{A}[1]^{\otimes d} \otimes \mathcal{P}, \tag{2.42}$$

and the differential acts as

$$\partial(q \otimes \underbrace{t \otimes \cdots \otimes t}_{d} \otimes p) = \sum_{i=0}^{d} \left( (-1)^{|m|} \delta_i^{\mathcal{Q}}(q) \otimes \underbrace{t \otimes \cdots \otimes t}_{d-i} \otimes p + q \otimes \underbrace{t \otimes \cdots \otimes t}_{d-i} \otimes \delta_i^{\mathcal{P}}(p) \right). \tag{2.43}$$

The tensor product $\mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}$ is functorial in the sense that if $F = \{F^d\} : \mathcal{P}_0 \to \mathcal{P}_1$ is a pre-morphism of $S^1$-complexes, then there are induced maps

$$F_\# : \mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}_0 \to \mathcal{Q} \otimes_{S^1}^{\mathbb{L}} \mathcal{P}_1, {}_\#F : \mathcal{P}_0 \otimes_{S^1}^{\mathbb{L}} \mathcal{Q} \to \mathcal{P}_1 \otimes_{S^1}^{\mathbb{L}} \mathcal{Q} \tag{2.44}$$

given by

$$F_\#(q \otimes \underbrace{t \otimes \cdots \otimes t}_{d} \otimes p) = \sum_{j=0}^{d} q \otimes \underbrace{t \otimes \cdots \otimes t}_{d-j} \otimes F^j(p) \tag{2.45}$$

and

$${}_\#F(p \otimes \underbrace{t \otimes \cdots \otimes t}_{d} \otimes q) = \sum_{j=0}^{d} (-1)^{\deg(F) \cdot |q|} F^j(p) \otimes \underbrace{t \otimes \cdots \otimes t}_{d-j} \otimes q, \tag{2.46}$$

which are chain maps if $F$ is closed.

**Proposition 2.6.1** ([43], Proposition 1)**.** *Let $F : \mathcal{P} \to \mathcal{P}'$ be a quasi-isomorphism of $S^1$-complexes, then we have induced quasi-isomorphisms between* hom *spaces and tensor products, i.e. we have quasi-isomorphisms*

$$F\circ : R\mathrm{Hom}_{S^1}(\mathcal{P}', \mathcal{Q}) \cong R\mathrm{Hom}_{S^1}(\mathcal{P}, \mathcal{Q}), \circ F : R\mathrm{Hom}_{S^1}(\mathcal{Q}, \mathcal{P}) \cong R\mathrm{Hom}_{S^1}(\mathcal{Q}, \mathcal{P}'), \tag{2.47}$$

$$_{\#}F : \mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P} \cong \mathcal{Q} \otimes^{\mathbb{L}}_{S^1} \mathcal{P}', F_{\#} : \mathcal{P} \otimes^{\mathbb{L}}_{S^1} \mathcal{Q} \cong \mathcal{P}' \otimes^{\mathbb{L}}_{S^1} \mathcal{Q}. \tag{2.48}$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be two $S^1$-complexes, and $f : \mathcal{P} \to \mathcal{Q}$ a chain map. We say that a homomorphism $F = \left\{F^d\right\}$ from $\mathcal{P}$ to $\mathcal{Q}$ an $S^1$-*equivariant enhancement* of $f$ if $[F^0] = [f]$. In particular, $F$ has degree $\deg(f)$.

Finally, we notice that if $\mathcal{P}$ and $\mathcal{Q}$ are $S^1$-complexes, then their (linear) tensor product $\mathcal{P} \otimes \mathcal{Q}$ can also be equipped with an $S^1$-complex structure with

$$\delta_k^{\mathcal{P}\otimes\mathcal{Q}}(p \otimes q) := (-1)^{|q|}\delta_k^{\mathcal{P}}(p) \otimes q + p \otimes \delta_k^{\mathcal{Q}}(q). \tag{2.49}$$

We call this $S^1$-action on $\mathcal{P} \otimes \mathcal{Q}$ the *diagonal $S^1$-action*.

**Definition 2.6.2** ([43], Definitions 3 and 4)**.** *The homotopy orbit complex of $\mathcal{P}$ is the derived tensor product*

$$\mathcal{P}_{hS^1} := \mathbb{K} \otimes^{\mathbb{L}}_{S^1} \mathcal{P}. \tag{2.50}$$

*The homotopy fixed point complex of $\mathcal{P}$ is the chain complex of morphisms*

$$\mathcal{P}^{hS^1} := R\mathrm{Hom}_{S^1}(\mathbb{K}, \mathcal{P}). \tag{2.51}$$

*The Tate complex of $\mathcal{P}$ is defined as the localization of $\mathcal{P}^{hS^1}$ at $u$*

$$\mathcal{P}^{Tate} := \mathcal{P}^{hS^1} \otimes_{\mathbb{K}[u]} \mathbb{K}[u, u^{-1}], \tag{2.52}$$

*where $u$ is a formal variable with $|u| = 2$.*

Let $F : \mathcal{P} \to \mathcal{Q}$ be a homomorphism of $S^1$-complexes, it induces chain maps

$$F^{hS^1} : \mathcal{P}^{hS^1} \to \mathcal{Q}^{hS^1}, F_{hS^1} : \mathcal{P}_{hS^1} \to \mathcal{Q}_{hS^1}, F^{Tate} : \mathcal{P}^{Tate} \to \mathcal{Q}^{Tate}. \tag{2.53}$$

When $F$ is a quasi-isomorphism (of $S^1$-complexes), $F^{hS^1}$, $F_{hS^1}$ and $F^{Tate}$ are quasi-isomorphisms (of chain complexes).

**Proposition 2.6.2** ([43], Proposition 2)**.** *If $F : \mathcal{P} \to \mathcal{Q}$ is a homomorphism between $S^1$-complexes, then the various induced maps in (2.53) intertwine all of the long exact sequences for equivariant homology groups of $\mathcal{P}$ with those for $\mathcal{Q}$.*

For instance, we have the Gysin exact triangle

$$\mathcal{P}_{hS^1} \to \mathcal{P}_{hS^1}[2] \to \mathcal{P} \xrightarrow{[1]} \tag{2.54}$$

Taking $\mathcal{P} = CH_*(\mathcal{A})$ to be the Hochschild chain complex of some strictly unital $A_\infty$-algebra $\mathcal{A}$ recovers Connes' long exact sequence (1.34). If $F : \mathcal{P} \to \mathcal{Q}$ is an $S^1$-complex homomorphism, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{P}_{hS^1} \arrow[r] \arrow[d] & \mathcal{P}_{hS^1}[2] \arrow[r] \arrow[d] & \mathcal{P} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\mathcal{Q}_{hS^1} \arrow[r] & \mathcal{Q}_{hS^1}[2] \arrow[r] & \mathcal{Q} \arrow[r, "{[1]}"] & {}
\end{tikzcd} \tag{2.55}$$

The structure of an $S^1$-complex $(\mathcal{P}, \{\delta_j\}_{j\geqslant 0})$ admits an alternative description by implementing the $u$-linear model. Let $u$ be a formal variable of degree 2, consider the $u$-adically completed tensor product

$$\mathcal{P}[[u]] := \mathcal{P}\widehat{\otimes}_{\mathbb{K}}\mathbb{K}[[u]] \tag{2.56}$$

in the category of graded vector spaces. An $S^1$-complex can be equivalently formulated as a graded $\mathbb{K}$-vector space $\mathcal{P}$ equipped with a map $\delta_{eq} : \mathcal{P} \to \mathcal{P}[[u]]$ of total degree 1 defined by

$$\delta_{eq} := \sum_{j=0}^{\infty} \delta_i u^i, \tag{2.57}$$

which satisfies $\delta_{eq}^2 = 0$. The map $\delta_{eq}$ is called an *equivariant differential*, note that it extends $u$-linearly to a map $\mathcal{P}[[u]] \to \mathcal{P}[[u]]$, which we will still denote by $\delta_{eq}$.

The *(positive) $S^1$-equivariant homology*, *negative $S^1$-equivariant homology* and *periodic $S^1$-equivariant homology* are defined respectively as homologies of the following complexes:

$$\mathcal{P}_{hS^1} := \left(\mathcal{P}((u))/u\mathcal{P}[[u]], \delta_{eq}\right), \tag{2.58}$$

$$\mathcal{P}^{hS^1} := \left(\mathcal{P}[[u]], \delta_{eq}\right), \tag{2.59}$$

$$\mathcal{P}^{Tate} := \left(\mathcal{P}((u)), \delta_{eq}\right). \tag{2.60}$$

## 2.7 Non-unital Hochschild chain complex

Our exposition here follows Section 3.1 of [43] closely. Let $\mathcal{A}$ be an $A_\infty$-algebra over the semisimple ring $\Bbbk := \bigoplus_{i=1}^{r} \mathbb{K}e_i$. For the usual Hochschild chain complex $CH_*(\mathcal{A})$ to be a strict $S^1$-complex in the sense of Definition 2.6.1, one needs to assume that $\mathcal{A}$ is strictly unital. However, in the geometric context, the Fukaya category is in general only cohomologically unital, we therefore need a replacement of the usual Hochschild chain complex, so that it possess the structure of a strict $S^1$-complex and is quasi-isomorphic to the usual Hochschild complex $CH_*(\mathcal{A})$. This construction, known as the *non-unital Hochschild chain complex*, will be recalled below.

Let $\mathcal{A}$ be a cohomologically unital $A_\infty$-algebra over $\Bbbk$. As a graded vector space, the non-unital Hochschild chain complex consists of two copies of the ordinary Hochschild chain complex, with the grading of the second copy shifted down by

1, i.e.

$$CH_*^{nu}(\mathcal{A}) := CH_*(\mathcal{A}) \oplus CH_*(\mathcal{A})[1]. \tag{2.61}$$

With respect to the decomposition (2.61), elements in the complex $CH_*^{nu}(\mathcal{A})$ can be written as $\check{\alpha} + \hat{\beta}$, where $\check{\alpha} \in CH_*(\mathcal{A})$ and $\hat{\beta} \in CH_*(\mathcal{A})[1]$. As a convention, we will refer to the left factor in $CH_*^{nu}(\mathcal{A})$ as the *check factor* and the right factor in $CH_*^{nu}(\mathcal{A})$ as the *hat factor*.

The differential $b^{nu}$ on the complex $CH_*^{nu}(\mathcal{A})$ can therefore be expressed as a block matrix

$$b^{nu} := \begin{bmatrix} b & d_{\wedge\vee} \\ 0 & b' \end{bmatrix} \tag{2.62}$$

where $b$ is the usual Hochschild differential on the check factor $CH_*(\mathcal{A})$, $b'$ is the *bar differential* on the hat factor defined by

$$\begin{aligned} b'(\hat{\beta}) = &\sum (-1)^{\maltese_1^s} x_d \otimes \cdots \otimes x_{s+j+1} \otimes \mu_{\mathcal{A}}^j(x_{s+j} \otimes \cdots x_{s+1}) \otimes x_s \otimes \cdots \otimes x_1 \\ &+ \sum (-1)^{\maltese_1^{d-j}} \mu_{\mathcal{A}}^j(x_d \otimes \cdots \otimes x_{d-j+1}) \otimes x_{d-j} \otimes \cdots \otimes x_1 \end{aligned} \tag{2.63}$$

where $\hat{\beta} = x_d \otimes \cdots \otimes x_1$, and $d_{\wedge\vee} : CH_*(\mathcal{A})[1] \to CH_*(\mathcal{A})$ is defined by

$$d_{\wedge\vee}(\hat{\beta}) := (-1)^{\maltese_2^d + ||x_1|| \cdot \maltese_2^d + 1} x_1 \otimes x_d \otimes \cdots \otimes x_2 + (-1)^{\maltese_1^{d-1}} x_d \otimes \cdots \otimes x_1. \tag{2.64}$$

In the above, we have followed the convention of [89], in particular the symbol

$$\maltese_i^j := \sum_{k=i}^{j} ||x_k|| \tag{2.65}$$

is used to abbreviate the signs, where $||x_k|| := |x_k| - 1$ is the reduced grading.

**Remark 2.7.1.** *The idea of non-unital Hochschild complex also appears in the geometric context, for example, in the Legendrian surgery description of the symplectic cohomology [16].*

It follows from Lemma 3 of [43] that the natural inclusion $CH_*(\mathcal{A}) \hookrightarrow CH_*^{nu}(\mathcal{A})$ is a quasi-isomorphism. Associated to $CH_*^{nu}(\mathcal{A})$ there is a corresponding non-unital version of the (chain level representative of) Connes' operator $B^{nu} : CH_*^{nu}(\mathcal{A}) \to CH_*^{nu}(\mathcal{A})[1]$, which is defined explicitly by

$$\begin{aligned} &B^{nu}(x_k \otimes \cdots \otimes x_1, y_l \otimes \cdots \otimes y_1) := \\ &\sum_i (-1)^{\maltese_1^i \maltese_{i+1}^k + ||x_k|| + \maltese_1^k + 1}(0, x_i \otimes \cdots \otimes x_1 \otimes x_k \otimes \cdots \otimes x_{i+1}). \end{aligned} \tag{2.66}$$

Note that we can write $B^{nu} = s^{nu}N$, where

$$s^{nu}(x_k \otimes \cdots \otimes x_1, y_l \otimes \cdots \otimes y_1) := (-1)^{\maltese_1^k + ||x_k|| + 1}(0, x_k \otimes \cdots \otimes x_1), \tag{2.67}$$

and

$$N(x_k \otimes \cdots \otimes x_1) := (1 + \lambda + \cdots + \lambda^{k-1})(x_k \otimes \cdots \otimes x_1) \tag{2.68}$$

is the *norm* of the *cyclic permutation operator*

$$\lambda(x_k \otimes \cdots \otimes x_1) := (-1)^{||x_1|| \cdot \maltese_2^k + ||x_1|| + ||x_k||} x_1 \otimes x_k \otimes \cdots \otimes x_2. \tag{2.69}$$

One can verify that $(B^{nu})^2 = 0$ and $b^{nu}B^{nu} + B^{nu}b^{nu} = 0$, which shows that

**Lemma 2.7.1** ([43], Lemma 4)**.** *$CH_*^{nu}(\mathcal{A})$ is a strict $S^1$-complex.*

When $\mathcal{A}$ is strictly unital, there is an $S^1$-equivariant enhancement of the natural inclusion $CH_*(\mathcal{A}) \hookrightarrow CH_*^{nu}(\mathcal{A})$, which is a quasi-isomorphism of $S^1$-complexes. Again, one can package everything in the $u$-linear model, and define the equivariant differential on $CH_*^{nu}(\mathcal{A})$ as

$$b_{eq} := b^{nu} + uB^{nu}. \tag{2.70}$$

The positive, negative and periodic cyclic homologies of a cohomoloically unital $A_\infty$-algebra $\mathcal{A}$ are then defined respectively as the homologies of the following complexes:

$$CC_*(\mathcal{A}) := \left(CH_*^{nu}(\mathcal{A}) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]], b_{eq}\right), \tag{2.71}$$

$$CC_*^-(\mathcal{A}) := \left(CH_*^{nu}(\mathcal{A})\widehat{\otimes}_{\mathbb{K}}\mathbb{K}[[u]], b_{eq}\right), \tag{2.72}$$

$$CC_*^{per} := \left(CH_*^{nu}(\mathcal{A})\widehat{\otimes}_{\mathbb{K}}\mathbb{K}((u)), b_{eq}\right). \tag{2.73}$$

# Chapter 3

# Formality and split-generation

By assuming the validity of Theorem 1.2.1, we study the symplectic topology of the Liouville 6-manifolds $M_{p,q,r}$ and prove Corollaries 1.2.1, 1.2.2 and 1.2.3. In this chapter, $\mathbb{K}$ will be an arbitrary field.

## 3.1 Suspension of a Lefschetz fibration

We start by recalling Seidel's work on suspending Lefschetz fibrations [92]. Let $E$ be a $2n$-dimensional Liouville manifold and let $\pi : E \to \mathbb{C}$ be an exact symplectic Lefschetz fibration with smooth fiber $M$, which is also a Lioiuville manifold. Assume that $c_1(M) = 0$. The *suspension*

$$\pi^\sigma : E^\sigma := E \times \mathbb{C} \to \mathbb{C} \tag{3.1}$$

of the Lefschetz fibration $\pi$ is defined as $\pi^\sigma(x, y) = \pi(x) + y^2$, where $x \in E$ and $y \in \mathbb{C}$. Note in particular that $\pi^\sigma$ is still a Lefschetz fibration. Denote by $M^\sigma$ a smooth fiber of $\pi^\sigma$, which is again a $2n$-dimensional Liouville manifold with $c_1(M^\sigma) = 0$, so we have two well-defined $\mathbb{Z}$-graded $A_\infty$-categories: the Fukaya category $\mathcal{F}(M)$ of the smooth fiber of $\pi$, and the Fukaya category $\mathcal{F}(M^\sigma)$ of the smooth fiber of $\pi^\sigma$. We describe here the algebraic construction of Seidel [92], which, when applied to geometry, describes a full $A_\infty$-subcategory $\mathcal{V}(M^\sigma) \subset \mathcal{F}(M^\sigma)$ in terms of the $A_\infty$-category $\mathcal{A}(\pi)$ associated to the Lefschetz fibration $\pi$.

Let $\mathcal{B}$ be a $\mathbb{Z}$-graded, strictly unital, proper $A_\infty$-category, defined over any field $\mathbb{K}$, fix a set $\{S_1, \ldots, S_k\}$ of non-trivial objects of $\mathcal{B}$, which means that $\hom_{\mathcal{B}}(S_i, S_i)$ is never acyclic for $1 \leqslant i \leqslant k$. Let $\mathcal{A} \subset \mathcal{B}$ be the directed $A_\infty$-subcategory with the same objects as $\mathcal{B}$ but whose morphism spaces are set to be

$$\hom_{\mathcal{A}}(S_i, S_j) = \begin{cases} \hom_{\mathcal{B}}(S_i, S_j) & i < j \\ \mathbb{K} \cdot e_{S_i} & i = j \\ 0 & i > j \end{cases} \tag{3.2}$$

The $A_\infty$-structure on $\mathcal{A}$ is defined to be the restriction of that of $\mathcal{B}$.

Let $C\ell_2(\mathcal{B})$ be another $A_\infty$-category with objects

$$(S_1^-, \ldots, S_k^-, S_1^+, \ldots, S_k^+), \tag{3.3}$$

where $S_i^+$ is a copy of $S_i$, while $S_i^-$ is a shifted copy $S_i[1]$. As the simplest instance of $A_\infty$-Morita equivalence, $C\ell_2(\mathcal{B})$ is quasi-isomorphic to $\mathcal{B}$. Let $\mathcal{C} \subset C\ell_2(\mathcal{B})$ be the associated directed $A_\infty$-subcategory. Schematically we have

$$\mathcal{C} = \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{B}[-1] & \mathcal{A} \end{bmatrix} \subset C\ell_2(\mathcal{B}) = \begin{bmatrix} \mathcal{B} & \mathcal{B}[1] \\ \mathcal{B}[-1] & \mathcal{B} \end{bmatrix}. \tag{3.4}$$

Finally, we introduce a third $A_\infty$-category $\mathcal{B}^\sigma$ with objects $(S_1^\sigma, \ldots, S_k^\sigma)$. This is the full $A_\infty$-subcategory of $\mathcal{C}^{tw}$ consisting of the twisted complexes

$$S_i^\sigma = Cone(S_i^-[-1] \xrightarrow{e_{S_i}} S_i^+) = \left( S_i^- \oplus S_i^+, \delta_{S_i^\sigma} = \begin{bmatrix} 0 & 0 \\ e_{S_i} & 0 \end{bmatrix} \right). \tag{3.5}$$

Under the assumption that each $S_i$ is simple, i.e.

$$H^0(\hom_{\mathcal{B}}(S_i, S_i)) = \mathbb{K}[e_{S_i}] \tag{3.6}$$

for $i = 1, \dots, k$, the directed $A_\infty$-category $\mathcal{A}$, together with the quasi-isomorphism class of the $\mathcal{A}$-bimodule $\mathcal{B}$ determine the $A_\infty$-category $\mathcal{B}^\sigma$ up to quasi-isomorphism.

The $A_\infty$-category $\mathcal{B}^\sigma$ is called the *algebraic suspension* of $\mathcal{B}$. Let $\mathcal{A}^\sigma \subset \mathcal{B}^\sigma$ be its associated directed $A_\infty$-subcategory, it is easy to see that $\mathcal{A}^\sigma$ is quasi-isomorphic to $\mathcal{A}$. The main result of [92] is a description of the algebraic suspension $\mathcal{B}^\sigma$ in terms of the pairing $(\mathcal{A}, \mathcal{B})$.

**Lemma 3.1.1** ([92], Lemma 4.2)**.** *Assume that (3.6) holds, and as an $\mathcal{A}$-bimodule, $\mathcal{B}$ is quasi-isomorphic to $\mathcal{A} \oplus (\mathcal{B}/\mathcal{A})[-1]$, then the $A_\infty$-category $\mathcal{B}^\sigma$ is quasi-isomorphic to the trivial extension constructed from $\mathcal{A}$ and the $\mathcal{A}$-bimodule $(\mathcal{B}/\mathcal{A})[-1]$.*

Geometrically, the construction above can be applied to the pairing

$$(\mathcal{A}, \mathcal{B}) = (\mathcal{A}(\pi), \mathcal{V}(M)), \tag{3.7}$$

where $\mathcal{V}(M) \subset \mathcal{F}(M)$ is the full $A_\infty$-subcategory which consists of vanishing cycles $V_1, \dots, V_k \subset M_*$, where $M_* \cong M$ is the fiber of $\pi$ over some chosen base point $* \in \mathbb{C}$. Starting from a distinguished basis of vanishing cycles $V_1, \dots, V_k$, the Lagrangian spheres $V_1^\sigma, \dots, V_k^\sigma \subset M_*^\sigma$ can be described as double branched covers of the corresponding basis of Lefschetz thimbles $\Delta_1, \dots, \Delta_k$ whose vanishing paths share the common end point $*$. In particular, $\partial\Delta_i = V_i$. Denote by $\mathcal{V}(M^\sigma) \subset \mathcal{F}(M^\sigma)$ the full $A_\infty$-subcategory formed by $V_1^\sigma, \dots, V_k^\sigma$, the algebraic constructions above can be translated into geometry via the following:

**Proposition 3.1.1** ([92], Lemma 6.3). *Let $\mathbb{K}$ be a field with* $\mathrm{char}(\mathbb{K}) \neq 2$. *There is a quasi-isomorphism between $A_\infty$-categories over $\mathbb{K}$:*

$$\mathcal{V}(M^\sigma) \cong \mathcal{V}(M)^\sigma. \tag{3.8}$$

The weak Calabi-Yau property of the $A_\infty$-category $\mathcal{V}(M)$ implies the existence of a quasi-isomorphism

$$\mathcal{V}(M)/\mathcal{A}(\pi) \cong \mathcal{A}(\pi)^\vee[-n+1] \tag{3.9}$$

between $\mathcal{A}(\pi)$-bimodules. Note that this is *strictly weaker* than the assumption in Lemma 3.1.1, which requires the existence of a quasi-isomorphism

$$\mathcal{V}(M) \cong \mathcal{A}(\pi) \oplus \mathcal{A}(\pi)^\vee[-n+1] \tag{3.10}$$

between $\mathcal{A}(\pi)$-bimodules. Because of this, in general it is not true that the $A_\infty$-category $\mathcal{V}(M^\sigma)$ is quasi-equivalent to the trivial extension $\mathcal{A}(\pi) \oplus \mathcal{A}(\pi)^\vee[-n]$.

**Remark 3.1.1.** *In a recent paper [72], Lekili and Ueda prove that the Fukaya $A_\infty$-algebras of certain distinguished bases of vanishing cycles in the Milnor fibers of weighted homogeneous singularities with a conditions on the weights are non-formal. In particular, the Milnor fibers of the surface singularities of the form*

$$x^p + y^q + z^2 = 0, \text{ with } \frac{1}{p} + \frac{1}{q} < \frac{1}{2} \tag{3.11}$$

*provide explicit examples of Liouville manifolds $M$ which satisfy (3.9) but not (3.10).*

In the other direction, it is proved by Seidel in [92] that the condition (3.10) is satisfied for fibers $M^\sigma$ obtained by once suspensions, namely there is a quasi-isomorphism

$$\mathcal{V}(M^\sigma) \cong \mathcal{A}(\pi) \oplus \mathcal{A}(\pi)^\vee[-n] \tag{3.12}$$

as $\mathcal{A}(\pi)$-bimodules. Denote by $\mathcal{V}(M^{\sigma\sigma})$ the $A_\infty$-category of vanishing cycles $V_1^{\sigma\sigma}, \ldots, V_k^{\sigma\sigma} \subset M^{\sigma\sigma}$ of the double suspension $\pi^{\sigma\sigma} : E \times \mathbb{C}^2 \to \mathbb{C}$. As a corollary of Lemma 3.1.1, we have the following:

**Corollary 3.1.1** ([92], Corollary 6.5)**.** *$\mathcal{V}(M^{\sigma\sigma})$ is quasi-equivalent to the trivial extension $\mathcal{A}(\pi) \oplus \mathcal{A}(\pi)^\vee[-n-1]$. In particular, its endomorphism algebra $\mathcal{V}_{M^{\sigma\sigma}}$ is formal as an $A_\infty$-algebra over $\Bbbk$ if the directed $A_\infty$-algebra $\mathcal{A}_\pi$ is formal.*

## 3.2 Generalized Eilenberg-Moore equivalence

For our purposes we need to recall a generalized version of the quasi-equivalence (1.4) established by Ekholm-Lekili [36].

Fix a finite set $\Gamma$. Let $\overline{M}_{-\Lambda}$ be a $2n$-dimensional Weinstein domain, with its Liouville form denoted by $\theta_M$. For each $v \in \Gamma$, let $\overline{L}_v \subset \overline{M}_{-\Lambda}$ be an oriented, connected, *Spin* Lagrangian submanifold with vanishing Maslov class. Its boundary $\partial\overline{L}_v \subset \partial\overline{M}_{-\Lambda}$ defines a Legendrian submanifold $\Lambda_v$ with respect to the contact structure on $\partial\overline{M}_{-\Lambda}$ defined by the restriction of $\theta_M$. Moreover, we assume that different $\overline{L}_v$'s intersect with each other transversely, and the intersections happen only in the interior of $\overline{M}_{-\Lambda}$. In particular, the Legendrian submanifolds $\Lambda_v$'s are disjoint from each other in $\partial\overline{M}_{-\Lambda}$, together they form a link $\Lambda := \bigsqcup_{v\in\Gamma} \Lambda_v$. Attaching $n$-handles to $\overline{M}_{-\Lambda}$ along the Legendrian link $\Lambda$ gives rise to a new Weinstein domain $\overline{M}$. Note that $\overline{M}$ contains a set of closed Lagrangian submanifolds $\{L_v\}_{v\in\Gamma}$, which are unions of $\overline{L}_v$ with the Lagrangian core discs of the Weinstein handles attached along $\Lambda$. Define

$$\mathcal{V}_M := \bigoplus_{v,w\in\Gamma} CF^*(L_v, L_w), \tag{3.13}$$

to be the Fukaya $A_\infty$-algebra of these Lagrangian submanifolds, which is well-defined and $\mathbb{Z}$-graded with our assumptions on the $\overline{L}_v$'s. This is an $A_\infty$-algebra over $\Bbbk$. For simplicity, we shall assume that $\mathcal{V}_M$ is strictly unital. If it is not, there is always a standard algebraic procedure which replaces it with a quasi-isomorphic $A_\infty$-algebra which is strictly unital (Lemma 2.1 of [89]). On the other hand, the Legendrian link $\Lambda \subset \partial\overline{M}_{-\Lambda}$ also has an associated dg algebra, the Chekanov-Eliashberg algebra $CE^*(\Lambda)$, which carries the structure of a dg algebra over $\Bbbk$, see Section 6.3 for its detailed definition in the special case when $\overline{M}_{-\Lambda}$ is the standard symplectic disc $D^6$. In general, it is generated as a $\mathbb{K}$-vector space by words of Reeb chords ending on $\Lambda$, i.e.

$$CE^*(\Lambda) := \bigoplus_{i=0}^{\infty} \mathbb{K}\langle\mathcal{R}\rangle^{\otimes i}, \tag{3.14}$$

where $\mathcal{R}$ is the set of Reeb chords. The differential is defined by counting $J$-holomorphic discs with boundary punctures in the symplectization $\mathbb{R}\times\partial\overline{M}_{-\Lambda}$, whose boundary components lie in the Lagrangian submanifold $\mathbb{R}\times\Lambda$, and whose punctures are asymptotic to the Reeb chords in $\mathcal{R}$. Since the union of the Lagrangian submanifolds $\bigcup_{v\in\Gamma}\overline{L}_v \subset \overline{M}_{-\Lambda}$ gives a filling of the Legendrian link $\Lambda$, there is an induced augmentation

$$\varepsilon_L : CE^*(\Lambda) \to \Bbbk, \tag{3.15}$$

which equips $CE^*(\Lambda)$ with the structure of an augmented dg algebra over $\Bbbk$. We have the following generalization of the Eilenberg-Moore equivalence (1.4):

**Theorem 3.2.1** ([36], Theorem 4)**.** *We have the following quasi-isomorphism*

$$R\mathrm{Hom}_{CE^*(\Lambda)}(\Bbbk,\Bbbk) \cong \mathcal{V}_M, \tag{3.16}$$

*where the left-hand side is computed with respect to the augmentation $\varepsilon_L$.*

## 3.3 Quiver algebras as Fukaya categories

In this section, we prove Corollary 1.2.1. As a by-product, it enables us to identify a full subcategory of the derived Fukaya category $D^{perf}\mathcal{F}(M_{p,q,r})$ with the derived category of perfect modules over an $A_\infty$-algebra associated to quiver with potential introduced in Section 2.1. Similar but more sophisticated results have been obtained by Smith in [105].

When $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\leqslant 1$, the Weinstein manifold $M_{p,q,r}$ is the Milnor fiber associated to the corresponding isolated singularity $t_{p,q,r}+w^2=0$. However, in this thesis we are also interested in the case when $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}>1$ and $p,q,r\geqslant 2$, where $M_{p,q,r}$ is no longer a Milnor fiber, but a generalized Milnor fiber in the sense of [60]. More precisely, this means that the polynomial $t_{p,q,r}+w^2$ on $\mathbb{C}^4$ has a finite number of isolated critical points, instead of a unique isolated critical point at the origin. By taking the intersection of a smooth fiber of $t_{p,q,r}+w^2:\mathbb{C}^4\to\mathbb{C}$ with a large ball $B^8\subset\mathbb{C}^4$, we still get a well-defined Weinstein domain whose completion is Weinstein deformation equivalent to $M_{p,q,r}$. Our considerations here will work for both of these cases.

In order to understand a full $A_\infty$-subcategory of the compact Fukaya categoty $\mathcal{F}(M_{p,q,r})$, we shall use the results assembled in the last two sections. Consider the Lefschetz fibration $\tilde{t}_{p,q,r}:\mathbb{C}^3\to\mathbb{C}$ defined as a Morsification of $t_{p,q,r}(x,y,z)$, see (1.20). Note that in our case, the smooth fiber of $\tilde{t}_{p,q,r}$ is symplectomorphic to the Milnor fiber $T_{p,q,r}\subset\mathbb{C}^3$ associated to the singularity $t_{p,q,r}(x,y,z)=0$. The suspension of $\tilde{t}_{p,q,r}$ is the Lefschetz fibration on $\mathbb{C}^4$ defined by

$$\tilde{f}_{p,q,r}(x,y,z,w):=\tilde{t}_{p,q,r}(x,y,z)+w^2. \tag{3.17}$$

Its smooth fiber is symplecomorphic to the Milnor fiber $M_{p,q,r}\subset\mathbb{C}^4$, whose symplectic topology is the main interest of this section.

Consider the full $A_\infty$-subcategory $\mathcal{V}(M_{p,q,r}) \subset \mathcal{F}(M_{p,q,r})$ whose objects are vanishing cycles $V_1, \dots, V_{p+q+r-1}$ in the Milnor fiber $M_{p,q,r}$. Since $M_{p,q,r}$ is a fiber of the suspension $\tilde{t}^{\sigma}_{p,q,r} : \mathbb{C}^4 \to \mathbb{C}$, the vanishing cycles in $M_{p,q,r}$ can be interpreted as double covers of the Lefschetz thimbles of $\tilde{t}_{p,q,r}$ branched along the vanishing cycles in the Milnor fiber $T_{p,q,r}$. Thus our notation for $\mathcal{V}(M_{p,q,r})$ is compatible with our previous convention, even when the manifold of interest is not necessarily a Milnor fiber. As a consequence, the $A_\infty$-category $\mathcal{V}(M_{p,q,r})$ is a *deformation* of the trivial extension

$$\mathcal{A}(\tilde{t}_{p,q,r}) \oplus \mathcal{A}(\tilde{t}_{p,q,r})^\vee[-3]. \tag{3.18}$$

To show that this deformation is trivial, we use Theorem 1.5.1 and the generalized Eilenberg-Moore equivalence (3.16) recalled in the last section. Take the Weinstein domain $\overline{M}_{-\Lambda}$ in Section 3.2 to be the standard symplectic ball $D^6$, and let $\Lambda = \Lambda_{p,q,r}$ be the link of Legendrian 2-spheres obtained in Proposition 4.3.1. It follows that $M = M_{p,q,r}$ and $\mathcal{V}_M = \mathcal{V}_{p,q,r}$. By Theorem 3.2.1, there is a quasi-isomorphism

$$R\mathrm{Hom}_{CE^*(\Lambda_{p,q,r})}(\mathbb{k}, \mathbb{k}) \cong \mathcal{V}_{p,q,r}. \tag{3.19}$$

On the other hand, our computation of the Chekanov-Eliashberg algebra in Theorem 1.5.1 gives the quasi-isomorphism

$$CE^*(\Lambda_{p,q,r}) \cong \Pi_3(\mathcal{A}_{p,q,r}). \tag{3.20}$$

In order to compute the Koszul dual of the 3-Calabi-Yau completion $\Pi_3(\mathcal{A}_{p,q,r})$ on the right-hand side above, it is more convenient to have an explicit model for it. the following Lemma enables us to identify $\Pi_3(\mathcal{A}_{p,q,r})$ with a Ginzburg algebra associated to some quiver with potential $(Q_{p,q,r}, w_{p,q,r})$.

**Lemma 3.3.1.** *The directed $A_\infty$-algebra $\mathcal{A}_{p,q,r}$ is quasi-isomorphic to a quiver algebra with global dimension no more than 2.*

*Proof.* It is proved by Keating in [60] that $\mathcal{A}_{p,q,r}$ is quasi-isomorphic to the endomorphism algebra of a tilting object for the hereditary category $Coh(\mathbb{P}^1_{p,q,r})$, where $Coh(\mathbb{P}^1_{p,q,r})$ is the abelian category of coherent sheaves on the weighted projective line $\mathbb{P}^1_{p,q,r}$. More precisely, it follows from the computation in [60] that $\mathcal{A}_{p,q,r}$ can be identified with the graded associative algebra $\mathbb{K}\overrightarrow{Q}_{p,q,r}/\overrightarrow{I}_{p,q,r}$ associated to the following directed quiver

$$\begin{tikzcd}
 & & \bullet_{P_1} \arrow[r, "x_1"] & \bullet_{P_2} \arrow[r, "x_2"] & \bullet\cdots\bullet \arrow[r, "x_{p-2}"] & \bullet_{P_{p-1}} \\
\bullet_A \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & \bullet_B \arrow[ur, "b_1"] \arrow[r, "b_2"] \arrow[dr, "b_3"] & \bullet_{Q_1} \arrow[r, "y_1"] & \bullet_{Q_2} \arrow[r, "y_2"] & \bullet\cdots\bullet \arrow[r, "y_{q-2}"] & \bullet_{Q_{q-1}} \\
 & & \bullet_{R_1} \arrow[r, "z_1"] & \bullet_{R_2} \arrow[r, "z_2"] & \bullet\cdots\bullet \arrow[r, "z_{r-2}"] & \bullet_{R_{r-1}}
\end{tikzcd} \tag{3.21}$$

with relations in $\overrightarrow{I}_{p,q,r}$ given by

$$b_2 \circ a_1 = 0, b_1 \circ a_2 = 0, b_3 \circ (a_1 - a_2) = 0. \tag{3.22}$$

It then follows that $\mathbb{K}\overrightarrow{Q}_{p,q,r}/\overrightarrow{I}_{p,q,r}$ is a *quasi-tilted algebra of canonical type*, see [73]. Quasi-tilted algebras are studied in [55], and they are characterized by having global dimension at most 2 and each indecomposable module having projective dimension at most 1 or injective dimension at most 1. □

By the above lemma, the conditions imposed on the quiver algebra $\mathcal{A}$ in Section 2.4 is satisfied for $\mathbb{K}\overrightarrow{Q}_{p,q,r}/\overrightarrow{I}_{p,q,r}$. Since by its construction recalled in Section 2.4, the Calabi-Yau completion $\Pi_3(\mathcal{A}_{p,q,r})$ is unchanged up to quasi-isomorphism by replacing $\mathcal{A}_{p,q,r}$ with the quiver algebra $\mathbb{K}\overrightarrow{Q}_{p,q,r}/\overrightarrow{I}_{p,q,r}$, which is quadratic by (3.22), using

Proposition 2.4.1 we can identify the 3-Calabi-Yau completion of $\mathcal{A}_{p,q,r}$ with the Ginzburg algebra $\mathcal{G}_{p,q,r} := \mathcal{G}(Q_{p,q,r}, w_{p,q,r})$ defined by the following quiver $Q_{p,q,r}$

$$
\begin{tikzcd}
 & & \bullet P_1 \arrow[r, "x_1"] \arrow[dll, bend right, "c_1"'] & \bullet P_2 \arrow[r, "x_2"] & \bullet \cdots \bullet \arrow[r, "x_{p-2}"] & \bullet P_{p-1} \\
\bullet A \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & \bullet B \arrow[ur, "b_1"] \arrow[r, "b_2"] \arrow[dr, "b_3"] & \bullet Q_1 \arrow[r, "y_1"] \arrow[ll, bend left, "c_2"] & \bullet Q_2 \arrow[r, "y_2"] & \bullet \cdots \bullet \arrow[r, "y_{q-2}"] & \bullet Q_{q-1} \\
 & & \bullet R_1 \arrow[r, "z_1"] \arrow[ull, bend left, "c_3"] & \bullet R_2 \arrow[r, "z_2"] & \bullet \cdots \bullet \arrow[r, "z_{r-2}"] & \bullet R_{r-1}
\end{tikzcd}
\tag{3.23}
$$

with potential

$$
w_{p,q,r} = a_1 b_2 c_2 + a_2 b_1 c_1 + a_1 b_3 c_3 - a_2 b_3 c_3. \tag{3.24}
$$

By Proposition 2.5.1,

$$
R\mathrm{Hom}_{\mathcal{G}_{p,q,r}}(\mathbb{k}, \mathbb{k}) \cong \mathcal{B}_{p,q,r}, \tag{3.25}
$$

where $\mathcal{B}_{p,q,r} := \mathcal{B}(Q_{p,q,r}, w_{p,q,r})$ is the compact 3-Calabi-Yau algebra associated to the same quiver with potential $(Q_{p,q,r}, w_{p,q,r})$.

**Lemma 3.3.2.** *There is a quasi-isomorphism*

$$
\mathcal{V}_{p,q,r} \cong \mathcal{B}_{p,q,r} \tag{3.26}
$$

*between $A_\infty$-algebras over $\mathbb{k}$.*

*Proof.* By our discussions in Section 2.5, one can use a quasi-isomorphic replacement of the Chekanov-Eliashberg dg algebra $CE^*(\Lambda_{p,q,r})$ when computing its Koszul dual. In order to make use of (3.25) to compute the left hand side of (3.19), besides the quasi-isomorphism (3.20), we need to show that the augmentation $\varepsilon_V : CE^*(\Lambda_{p,q,r}) \to \mathbb{k}$ induced by the Lagrangian fillings $\overline{V}_1, \ldots, \overline{V}_{p+q+r-1}$ of the Legendrian link $\Lambda_{p,q,r}$ by (half of the) vanishing cycles corresponds, under (3.20), to the trivial projection $\varepsilon : \mathcal{G}_{p,q,r} \to \mathbb{k}$, which is the augmentation that we used on the left hand side of (3.25) to compute the Koszul dual of $\mathcal{G}_{p,q,r}$. To see this, we need to refer to the explicit quasi-isomorphism (6.170) between the cellular dg algebra $\mathcal{C}(\Lambda_{2,2,2})$ and the Ginzburg algebra $\mathcal{G}_{2,2,2}$, which shows that the degree zero generators $a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3$ of the Ginzburg dg algebra $\mathcal{G}_{2,2,2}$ correspond geometrically to Reeb chords between different components of $\Lambda_{2,2,2}$ under the quasi-isomorphism (3.20). By definition, the image of these Reeb chords under $\varepsilon_V$ are zero, which shows that $(CE^*(\Lambda_{2,2,2}), \varepsilon_V)$ and $(\mathcal{G}_{2,2,2}, \varepsilon)$ are quasi-isomorphic as augmented dg algebras. The general case can be argued in a completely identical way, as the newly created degree 0 generators in $CE^*(\Lambda_{p,q,r})$ after attaching unknoted Legendrian spheres to $\Lambda_{2,2,2}$ are also Reeb chords between different components of $\Lambda_{p,q,r}$. $\square$

Since $\mathcal{B}_{p,q,r}$ is by construction the cyclic completion $\mathcal{A}_{p,q,r} \oplus \mathcal{A}^\vee_{p,q,r}[-3]$, we conclude that there is a quasi-isomorphism

$$\mathcal{V}_{p,q,r} \cong \mathcal{A}_{p,q,r} \oplus \mathcal{A}^\vee_{p,q,r}[-3], \tag{3.27}$$

which implies the formality of the $A_\infty$-algebra $\mathcal{V}_{p,q,r}$, by Proposition 2.3.1.

## 3.4 Split-generation

This section proves Corollary 1.2.2. Here we use the abbreviated notations $\mathcal{G}$ and $\mathcal{B}$ to denote the Ginzburg dg algebra and the Kontsevich-Soibelman's cyclic $A_\infty$-algebra associated to the same quiver with potential $(Q, w)$. By Proposition 2.5.1, when $w$ is cubic, $\mathcal{B} = E(\mathcal{G})$ is the (bigraded) Koszul dual of $\mathcal{G}$.

Recall that all the $A_\infty$-modules over $\mathcal{G}$ form a dg category $\mathcal{G}^{mod}$. There is a *Koszul duality functor*

$$\mathcal{K} : \mathcal{G}^{mod} \to \mathcal{B}^{mod} \tag{3.28}$$

defined by $R\mathrm{Hom}_{\mathcal{G}}(\Bbbk, \cdot)$. This is first introduced by Beilinson-Ginzburg-Soergel in [13] for Koszul algebras, its generalization in the context of $A_\infty$-Koszul duality is straightforward.

Denote by $D^{prop}(\mathcal{G})$ the derived category of proper $\mathcal{G}$-modules, namely those $\mathcal{G}$-modules whose cohomologies are finite dimensional; and by $D^{perf}(\mathcal{B})$ the derived category of perfect $\mathcal{B}$-modules, which is obtained by taking the split-closure of the homotopy category of the $A_\infty$-category of twisted complexes over $\mathcal{B}$, i.e. $H^0(Tw(\mathcal{B}))$.

**Proposition 3.4.1** ([11, 66]). *Assume that $H^*(\mathcal{G})$ is finite-dimensional in each fixed degree. The restriction of the Koszul duality functor $\mathcal{K}$ induces an equivalence*

$$D\mathcal{K} : D^{prop}(\mathcal{G}) \to D^{perf}(\mathcal{B}). \tag{3.29}$$

**Remark 3.4.1.** *Without the assumption that $H^*(\mathcal{G})$ is finite-dimensional in each degree, we have instead an equivalence*

$$D^{prop}(\widehat{\mathcal{G}}) \cong D^{perf}(\mathcal{B}), \tag{3.30}$$

*where $\widehat{\mathcal{G}}$ is the completed Ginzburg algebra. As we have seen in the proof of Proposition 2.5.1, when $H^*(\mathcal{G})$ is locally finite with respect to the total grading, there is a quasi-isomorphism $\mathcal{G} \cong \widehat{\mathcal{G}}$.*

*The bigraded version of the Koszul duality functor $\mathcal{K}$ induces an equivalence between $D^{prop}(\mathcal{G})$ and $D^{perf}(\mathcal{B})$ whenever $\mathcal{G}$ is Adams connected as a bigraded $A_\infty$-algebra, see Theorem B of [77]. Although the Adams connectedness condition is satisfied for any Ginzburg algebra $\mathcal{G}$ defined by a quiver $Q$ with cubic potential $w$, this version of derived equivalence cannot be used to prove the split-generation of the compact Fukaya category $\mathcal{F}(M_{p,q,r})$ by vanishing cycles. This is because the derived category of bigraded $A_\infty$-modules over the bigraded $A_\infty$-algebra $\mathcal{G}$ or $\mathcal{B}$ is in general unrelated to the corresponding singly graded version.*

Another ingredient which is relevant for the proof of Corollary 1.2.2 is the split-generation of the wrapped Fukaya category by Lagrangian cocores, which should follow by combing the work of Bourgeois-Ekholm-Eliashberg [16] on Legendrian surgery with Abouzaid's geometric generation criterion [2]. For convenience, we state the following generation result, which appears in the recent work [23]. A more general statement, which takes into account also Weinstein domains with stops, is proved in [47].

**Theorem 3.4.1** ([23, 47])**.** *Let $M$ be a Weinstein manifold, which can be realized as the result of Weinstein handle attachment to $D^{2n}$. Denote by $L_1, \dots, L_k$ the Lagrangian cocore discs in the $n$-handles, the wrapped Fukaya category $\mathcal{W}(M)$ over any field $\mathbb{K}$ is generated by $L_1, \dots, L_k$.*

We are now prepared to prove Corollary 1.2.2 by combining the facts stated above. In our specific setting, the Milnor fiber $M_{p,q,r}$ can be constructed by attaching $(p+q+r-1)$ 3-handles along $\Lambda_{p,q,r}$ to the standard symplectic ball $D^6$, see Figure 4.5. Recall that $\mathcal{W}_{p,q,r}$ is the endomorphism algebra of the Lagrangian cocores $L_1, \cdots, L_{p+q+r-1}$ in $M_{p,q,r}$.

Since we have proved the quasi-isomorphism $\mathcal{V}_{p,q,r} \cong \mathcal{B}_{p,q,r}$ in Section 3.3, and it follows from (1.22) and Theorem 1.2.1 that $\mathcal{W}_{p,q,r} \cong \mathcal{G}_{p,q,r}$. By Proposition 2.5.1, there is Koszul duality between the $A_\infty$-algebras $\mathcal{V}_{p,q,r}$ and $\mathcal{W}_{p,q,r}$, namely

$$E(\mathcal{V}_{p,q,r}) \cong \mathcal{W}_{p,q,r}, E(\mathcal{W}_{p,q,r}) \cong \mathcal{V}_{p,q,r}, \tag{3.31}$$

where $\mathcal{V}_{p,q,r}$ and $\mathcal{W}_{p,q,r}$ above are equipped with bigradings which coincide with the ones on the quiver algebras $\mathcal{B}_{p,q,r}$ and $\mathcal{G}_{p,q,r}$ described in the proof of Proposition 2.5.1. However, for geometric applications, we have to get rid of the double gradings and obtain a version of Koszul duality between $\mathcal{V}_{p,q,r}$ and $\mathcal{W}_{p,q,r}$ as $\mathbb{Z}$-graded $A_\infty$-algebras. This is possible by Proposition 2.5.2 and the following lemma.

**Lemma 3.4.1.** *Let $p \geqslant 2, q \geqslant 2, r \geqslant 2$. $H^*(\mathcal{G}_{p,q,r})$ is finite-dimensional in each fixed degree.*

*Proof.* From (3.23), we see that the only cycles of arrows in the quiver $Q_{p,q,r}$ are of the form $a_ib_jc_j$, where $i = 1,2$ and $j = 1,2,3$. To show that they vanish in the cohomology algebra $H^*(\mathcal{G}_{p,q,r})$, we need to take into account the relations coming from the differentials of the potential $w_{p,q,r}$. From $\partial w_{p,q,r}/\partial c_1$ and $\partial w_{p,q,r}/\partial c_2$, we see that $a_1b_2 = a_2b_1 = 0$ in $H^*(\mathcal{G}_{p,q,r})$, so the only possible non-zero cycles in the cohomology algebra are $a_1b_1c_1$, $a_1b_3c_3$, $a_2b_2c_2$ and $a_2b_3c_3$. From $\partial w_{p,q,r}/\partial a_1$ and $\partial w_{p,q,r}/\partial a_2$, one gets the relations

$$b_2c_2 + b_3c_3 = 0, b_1c_1 - b_3c_3 = 0 \tag{3.32}$$

in $H^*(\mathcal{G}_{p,q,r})$, so we are reduced to show that $a_1b_1c_1 = a_2b_2c_2 = 0$. From $\partial w_{p,q,r}/\partial c_3$, we get $a_1b_3 = a_2b_3$, so it suffices to prove that $a_1b_1c_1 = 0$. But from (3.32) we get

$$a_1b_1c_1 = a_1b_3c_3 = -a_1b_2c_2 = 0. \tag{3.33}$$

□

The above lemma enables us to apply Proposition 3.4.1, from which we get an equivalence

$$D^{prop}(\mathcal{G}_{p,q,r}) \cong D^{perf}(\mathcal{B}_{p,q,r}). \tag{3.34}$$

By Lemma 3.25 of [89] and Lemma 3.3.2, there is a derived equivalence $D^{perf}(\mathcal{B}_{p,q,r}) \cong D^{perf}(\mathcal{V}_{p,q,r})$. Similarly, Theorem 1.5.1 implies that $D^{perf}(\mathcal{G}_{p,q,r}) \cong D^{perf}(\mathcal{W}_{p,q,r})$. Since both of $\mathcal{G}_{p,q,r}$ and $\mathcal{W}_{p,q,r}$ are homologically smooth $A_\infty$-algebras, by restricting to the full subcategory of proper modules we get an equivalence

$$D^{prop}(\mathcal{G}_{p,q,r}) \cong D^{prop}(\mathcal{W}_{p,q,r}), \tag{3.35}$$

see Lecture 7 of [90]. We have thus interpreted the derived equivalence in Proposition 3.4.1 as an equivalence between derived categories of certain modules over Fukaya $A_\infty$-algebras:

$$D^{prop}(\mathcal{W}_{p,q,r}) \cong D^{perf}(\mathcal{V}_{p,q,r}). \tag{3.36}$$

By Theorem 3.4.1, we have an identification

$$D^{perf}(\mathcal{W}(M_{p,q,r})) \cong D^{perf}(\mathcal{W}_{p,q,r}). \tag{3.37}$$

It is proved by Ganatra in [44] that the wrapped Fukaya category of any Weinstein manifold is a homologically smooth $A_\infty$-category (in its general form, this fact depends on the results claimed in [16]), therefore the above equivalence restricts to an equivalence

$$D^{prop}(\mathcal{W}(M_{p,q,r})) \cong D^{prop}(\mathcal{W}_{p,q,r}). \tag{3.38}$$

Using this equivalence, (3.36) can be interpreted equivalently as an equivalence

$$D\mathcal{K} : D^{prop}(\mathcal{W}(M_{p,q,r})) \xrightarrow{\simeq} D^{perf}(\mathcal{V}(M_{p,q,r})). \tag{3.39}$$

On the other hand, since each object of $\mathcal{F}(M_{p,q,r})$ can be regarded as a proper $A_\infty$-module over $\mathcal{W}_{p,q,r}$ under the Yoneda functor

$$\mathcal{Y} : \mathcal{W}(M_{p,q,r}) \to \mathcal{W}^{mod}_{p,q,r}, \tag{3.40}$$

we get a fully faithful embedding

$$D\mathcal{J} : D^{perf}(\mathcal{F}(M_{p,q,r})) \hookrightarrow D^{prop}(\mathcal{W}(M_{p,q,r})). \tag{3.41}$$

Combining with the Koszul duality equivalence $D\mathcal{K}$, we obtain a fully faithful embedding

$$D\mathcal{K} \circ D\mathcal{J} : D^{perf}(\mathcal{F}(M_{p,q,r})) \hookrightarrow D^{perf}(\mathcal{V}(M_{p,q,r})), \tag{3.42}$$

which proves the split-generation of $\mathcal{F}(M_{p,q,r})$ by vanishing cycles.

**Remark 3.4.2.** *As a by-product of the above argument, we get the equivalence*

$$D^{prop}(\mathcal{W}(M_{p,q,r})) \cong D^{perf}(\mathcal{F}(M_{p,q,r})), \tag{3.43}$$

*which is expected to be true for a general Weinstein manifold $M$, namely when $\mathcal{F}(M)$ and $\mathcal{W}(M)$ are not necessarily related by $A_\infty$-Koszul duality. However, the fully faithfulness of the functor $D\mathcal{K}$ above relies on the fact that*

$$R\mathrm{Hom}_{\mathcal{V}_{p,q,r}}(\Bbbk, \Bbbk) \cong \mathcal{W}_{p,q,r} \tag{3.44}$$

*as $\mathbb{Z}$-graded $A_\infty$-algebras. Because of this, in general the generation of the wrapped Fukaya category $\mathcal{W}(M)$ by cocores does not lead to a split-generation result of the compact Fukaya category $\mathcal{F}(M)$.*

## 3.5 Quasi-dilation

In this section, $\mathcal{A}$ will be a special kind of a directed $A_\infty$-algebra over some semisimple ring $\Bbbk := \bigoplus_{1 \leqslant i \leqslant r} \mathbb{K}e_i$. To be precise, we make use of the following notion introduced in [24].

**Definition 3.5.1** ([24], Definition 5.10)**.** *A one-way algebra $\mathcal{A}$ is a finite dimensional algebra over $\Bbbk$ with a complete set $\{e_1, \cdots, e_r\}$ of orthogonal idempotents such that*

- *for $i \neq j$, if $e_i\mathcal{A}e_j \neq 0$, then $e_j\mathcal{A}e_i = 0$;*
- *for any idempotent $e_i$, we have $\dim_{\mathbb{K}}(e_i\mathcal{A}e_i) = 1$;*
- *$r > 1$ and $\mathcal{A}$ is indecomposable.*

Note that many of the known examples (say those studied in [61]) of the directed Fukaya categories $\mathcal{A}(\pi)$ associated to Lefschetz fibrations $\pi$ can actually be identified with one-way algebras. In particular, the directed $A_\infty$-algebras $\mathcal{A}_{p,q,r}$ encountered in Section 3.3 are one-way algebras.

From now on, assume that $\mathcal{A}$ is an one-way algebra. Denote by $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}^\vee[-n]$ the cyclic completion of $\mathcal{A}$. In particular, $\mathcal{B}$ is a proper $n$-Calabi-Yau algebra. Since $\mathcal{A}$ is formal, so is $\mathcal{B}$ by Proposition 2.3.1. Since we are actually working with $\mathbb{Z}$-graded associative algebras over $\mathbb{k}$, we will mainly take the algebraic, rather than the categorical point of view in this section. Recall that the Hochschild cochain complex $CH^*(\mathcal{B}, \mathcal{B})$ is defined to be

$$\hom_{\mathbb{k}}(T\overline{\mathcal{B}}, \mathcal{B}), \tag{3.45}$$

the space of $\mathbb{k}$-linear maps from the reduced tensor algebra of $\mathcal{B}$ to $\mathcal{B}$. Using the grading on $\mathcal{B}$, we get the decomposition

$$CH^d(\mathcal{B}, \mathcal{B}) = \bigoplus_{d=r+s} CH^r(\mathcal{B}, \mathcal{B}[s]), \tag{3.46}$$

where the right-hand side is the subspace of linear maps $\mathcal{B}^{\otimes r} \to \mathcal{B}$ of degree $s$. In particular, the Hochschild complex of a graded algebra is bigraded, and the corresponding Hochschild differential has bidegree $(1, 0)$.

Formality of the $A_\infty$-algebra $\mathcal{B}$ implies that there is a distinguished Hochschild cocycle

$$eu_{\mathcal{B}} \in CH^1(\mathcal{B}, \mathcal{B}) \tag{3.47}$$

defined by sending a homogeneous element $b$ with $|b| = i$ to $i \cdot b$. The fact that

$$|b_2 b_1| = |b_1| + |b_2| \tag{3.48}$$

then implies that $eu_{\mathcal{B}}$ is a derivation, namely $eu_{\mathcal{B}} \in HH^1(\mathcal{B}, \mathcal{B})$. We call $eu_{\mathcal{B}}$ the *Euler vector field*.

As a trivial extension, $\mathcal{B}$ is easily seen to be a ($\mathbb{Z}$-graded) symmetric algebra, whose non-degenerate inner product

$$\langle \cdot, \cdot \rangle : \mathcal{B} \otimes \mathcal{B} \to \Bbbk \tag{3.49}$$

is defined by

$$\langle b_1, b_2 \rangle = a_1 a_2^\vee + a_1^\vee a_2, \tag{3.50}$$

where $b_i = (a_i, a_i^\vee)$ with $a_i \in \mathcal{A}$ and $a_i^\vee \in \mathcal{A}^\vee[-n]$ for $i = 1, 2$. In more abstract terms, this inner product is induced from the proper Calabi-Yau structure on $\mathcal{B}$.

It is proved by Tradler in [110] that there is an algebraically defined BV operator

$$\Delta_{cyc} : HH^*(\mathcal{B}, \mathcal{B}) \to HH^{*-1}(\mathcal{B}, \mathcal{B}) \tag{3.51}$$

on the Hochschild cohomology of any cyclic $A_\infty$-algebra $\mathcal{B}$, which has bidegree $(-1, 0)$. For our purposes here, we shall omit the full formulae, and concentrate on its simplest piece on Hochschild cochains with pure degree $(1, 0)$:

$$\langle \Delta_{cyc}(c), b \rangle = \langle c(b), \mathrm{id}_{\mathcal{B}} \rangle, \tag{3.52}$$

where $b \in \mathcal{B}$, $c \in CH^1(\mathcal{B}, \mathcal{B}[0])$ and $\mathrm{id}_{\mathcal{B}}$ is the identity of $\mathcal{B}$.

Our main result in this section the following:

**Proposition 3.5.1.** *Let $\mathcal{B}$ be the $n$-cyclic completion of a one-way algebra $\mathcal{A}$ over $\Bbbk$, then*

$$\Delta_{cyc}\left(\frac{1}{n} eu_{\mathcal{B}}\right) = 1. \tag{3.53}$$

*Proof.* Let $a$ be a generator of $\mathcal{A}$. Since $\mathcal{A}$ is trivially graded, $|(a, 0)| = 0$ in $\mathcal{B}$. It follows from the definition of $\mathcal{B}$ that $|(0, a^\vee[-n])| = n$ in $\mathcal{B}$. Since $eu_{\mathcal{B}}$ has pure degree $(1, 0)$, from the definition of the BV operator $\Delta_{cyc}$ it follows that

$$\left\langle \Delta_{cyc}\left(\frac{1}{n} eu_{\mathcal{B}}\right), (0, a^\vee[-n]) \right\rangle = \left\langle \frac{1}{n} eu_{\mathcal{B}}(0, a^\vee[-n]), \mathrm{id}_{\mathcal{B}} \right\rangle, \tag{3.54}$$

By definition of the derivation $\frac{1}{n}eu_{\mathcal{B}}$ and the symmetric pairing $\langle\cdot,\cdot\rangle$ on a trivial extension algebra, the right-hand side of (3.54) is equal to

$$a^{\vee}[-n]\cdot \mathrm{id}_{\mathcal{A}}. \tag{3.55}$$

For any trivial extension algebra $\mathcal{B}$ of $\mathcal{A}$, it is easy to find that

$$Z(\mathcal{B}) = Z(\mathcal{A}) \ltimes Ann_{\mathcal{A}^{\vee}}(C(\mathcal{A})), \tag{3.56}$$

where $Z(\mathcal{A})$ and $Z(\mathcal{B})$ denote the (ungraded) centres of $\mathcal{A}$ and $\mathcal{B}$ respectively, $C(\mathcal{A}) \subset \mathcal{A}$ is the subspace of commutators, and

$$Ann_{\mathcal{A}^{\vee}}(V) := \{a^{\vee} \in \mathcal{A}^{\vee} | a^{\vee}(V) = 0\} \tag{3.57}$$

for any subspace $V \subset \mathcal{A}$, see [14].

The assumption that $\mathcal{A}$ is a one-way algebra implies easily that $Z(\mathcal{A}) \cong \mathbb{K}$. On the other hand, since the BV operator $\Delta_{cyc}$ has bidegree $(-1,0)$,

$$\Delta_{cyc}\left(\frac{1}{n}eu_{\mathcal{B}}\right) \in Z(\mathcal{B}) \subset HH^0(\mathcal{B},\mathcal{B}). \tag{3.58}$$

By (3.56), $\Delta_{cyc}\left(\frac{1}{n}eu_{\mathcal{B}}\right)$ can be expressed as $(\lambda\cdot\mathrm{id}_{\mathcal{A}}, \alpha^{\vee}[-n])$, where $\lambda \in \mathbb{K}$ and $\alpha^{\vee} \in Ann_{\mathcal{A}^{\vee}}(C(\mathcal{A}))$. Using this expression, the left-hand side of (3.54) becomes $(\lambda a^{\vee}[-n] + a\cdot\alpha^{\vee}[-n])\cdot\mathrm{id}_{\mathcal{A}}$, from which we get

$$\lambda a^{\vee}[-n] + a\cdot\alpha^{\vee}[-n] = a^{\vee}[-n]. \tag{3.59}$$

Since we may choose $a \in C(\mathcal{A})$ in (3.59), by definition of $\alpha^{\vee}$ we deduce $\lambda = 1$ in the above. It follows that $a\cdot\alpha^{\vee}[-n]$ for all $a \in \mathcal{A}$, which forces $\alpha^{\vee}[-n] = 0$. Thus we have proved that $\Delta_{cyc}(\frac{1}{n}eu_{\mathcal{B}}) = 1 \in HH^0(\mathcal{B},\mathcal{B})$. $\square$

**Remark 3.5.1.** *A related result has been obtained by Schedler in [87] for preprojective algebras $\Pi_Q$ associated to a non-Dynkin quiver $Q$. Since $\Pi_Q$ is 2-Calabi-Yau, the half-Euler vector field plays an important role there.*

Suppose a proper $n$-Calabi-Yau algebra $\mathcal{B}$ and a smooth $n$-Calabi-Yau algebra $\mathcal{G}$ are Koszul dual as $\mathbb{Z}$-graded $A_\infty$-algebras. It is first proved in [28] for classical Koszul Calabi-Yau algebras and later generalized to $A_\infty$-algebras in [57] that there is a BV algebra isomorphism

$$HH^*(\mathcal{B},\mathcal{B}) \cong HH^*(\mathcal{G},\mathcal{G}). \tag{3.60}$$

When $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}^\vee[-n]$ is the cyclic completion of a one-way algebra, Proposition 3.5.1 then implies the existence of a cohomology class in $HH^1(\mathcal{G},\mathcal{G})$, which we denote by $\frac{1}{n}eu_{\mathcal{G}}$, satisfying

$$\Delta_{CY}\left(\frac{1}{n}eu_{\mathcal{G}}\right) = 1, \tag{3.61}$$

where $\Delta_{CY}$ denotes the BV operator on $HH^*(\mathcal{G},\mathcal{G})$ defined by the Calabi-Yau structure on $\mathcal{G}$.

Back to our concrete set up, it follows from Lemma 3.4.1 that the Koszul duality between the $\mathbb{Z}$-graded $A_\infty$-algebras $\mathcal{B}_{p,q,r}$ and $\mathcal{G}_{p,q,r}$ implies the existence of a cohomology class $\frac{1}{3}eu_{p,q,r} \in HH^1(\mathcal{G}_{p,q,r},\mathcal{G}_{p,q,r})$ which is mapped to the identity by the BV operator $\Delta_{CY}$, where $\Delta_{CY}$ is defined using the smooth 3-Calabi-Yau structure on the Ginzburg dg algebra $\mathcal{G}_{p,q,r}$. On the other hand, although the quasi-isomorphism

$$\mathcal{W}_{p,q,r} \cong \mathcal{G}_{p,q,r} \tag{3.62}$$

induces an isomorphism between Hochschild cohomologies

$$HH^*(\mathcal{W}(M_{p,q,r}),\mathcal{W}(M_{p,q,r})) \cong HH^*(\mathcal{G}_{p,q,r},\mathcal{G}_{p,q,r}) \tag{3.63}$$

as Gerstenhaber algebras, it is not in general true that (3.63) preserves the underlying BV structures, as the smooth Calabi-Yau structure on the wrapped Fukaya category $\mathcal{W}(M_{p,q,r})$ constructed from symplectic geometry [43] may differ from the Calabi-Yau structure on the Ginzburg dg algebra $\mathcal{G}_{p,q,r}$. The variation of the Calabi-Yau structure changes the BV operator $\Delta_{CY}$ on $HH^*(\mathcal{G}_{p,q,r}, \mathcal{G}_{p,q,r})$ by the conjugate action of an invertible element

$$h \in HH^0(\mathcal{G}_{p,q,r}, \mathcal{G}_{p,q,r})^\times. \tag{3.64}$$

By the BV algebra isomorphism

$$SH^*(M_{p,q,r}) \cong HH^*(\mathcal{W}(M_{p,q,r}), \mathcal{W}(M_{p,q,r})) \tag{3.65}$$

proved by Ganatra [44], we conclude the existence of a quasi-dilation $(\frac{1}{3}eu_{p,q,r}, h) \in SH^1(M_{p,q,r}) \times SH^0(M_{p,q,r})^\times$.

**Remark 3.5.2.** *Another way of proving that $M_{p,q,r}$ admits a quasi-dilations over a field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$ is to use the Lefschetz fibrations*

$$\pi_{p,q,r} : M_{p,q,r} \to \mathbb{C} \tag{3.66}$$

*described in Section 4.1. Since the smooth fiber of $\pi_{p,q,r}$ is symplectomorphic to the 4-dimensional $D_4$ type Milnor fiber, one can apply Proposition 3.5.1 to the zig-zag algebra $B_T$ with the tree $T = D_4$ to see that the fiber of $\pi_{p,q,r}$ admits a quasi-dilation. Together with Seidel-Solomon's inductive argument based on Lefschetz fibration techniques [103], this implies that the total space $M_{p,q,r}$ also admits a quasi-dilation. However, this argument does not apply when $\mathbb{K} = \mathbb{Z}/2$, since in this case Koszul duality does not hold for $D_4$ Milnor fibers, see Theorem 14 of [40] for details.*

# Chapter 4

# From Lefschetz fibrations to Legendrian fronts

We describe a Lefschetz fibration, whose construction is essentially due to Keating [61], on $M_{p,q,r}$. With Casals-Murphy recipe [22], we are able to get the Legendrian front presentation of this fibration, which, after simplifications, gives the Legendrian attaching link $\Lambda_{p,q,r} \subset S^5$ of the Weinstein manifold $M_{p,q,r}$.

## 4.1 A Lefschetz fibration on $M_{p,q,r}$

We start by recalling a Lefschetz fibration on the Milnor fibers $T_{p,q,r} \subset \mathbb{C}^3$ (defined by (1.20)) constructed by Keating in [61]. The construction is divided into three steps.

First, consider the Milnor fiber of two variables $F_{p,q} \subset \mathbb{C}^2$ defined by the polynomial

$$g_{p,q}(x,y) := (x^{p-2} - y^2)(x^2 - \lambda y^{q-2}) - 1. \tag{4.1}$$

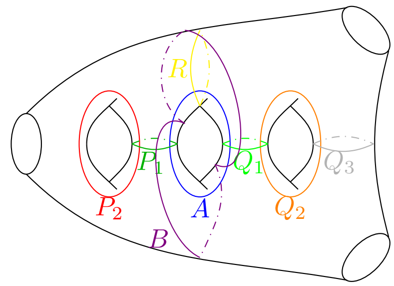


Figure 4.1: The vanishing cycles of $F_{3,4}$

Using A'Campo's method, the vanishing cycles of $F_{p,q}$ can be explicitly described by a divide of $\mathbb{R}^2$ induced by a real deformation $\tilde{g}_{p,q}$ of the polynomial $g_{p,q}$. After some mutations, we get a basis of vanishing cycles labelled by

$$A, B, P_1, \ldots, P_{p-1}, Q_1, \ldots, Q_{q-1}, R, \tag{4.2}$$

see Figure 4.1.

The second step is to produce a Lefschetz fibration on $T_{p,q,r}$. In this case, the Morsification of the polynomial

$$\tilde{g}_{p,q}(x, y) + z^r \tag{4.3}$$

has an associated a generalized Milnor fiber $\widehat{T}_{p,q,r}$, and $\overline{T}_{p,q,r}$ embeds in it as a Liouville sub-domain. There is a Lefschetz fibration $\hat{\pi}_T : \widehat{T}_{p,q,r} \to \mathbb{C}$ defined by a Morsification of $z^r$ in (4.3). To get a Lefschetz fibration on $T_{p,q,r}$, one chooses carefully a 1-parameter family of polynomials $m_t(x, y, z)$, such that $m_0 = \tilde{g}_{p,q} + \tilde{z}^r$, with $\tilde{z}^r$ being a Morsification of $z^r$, and the smooth affine surface defined by $m_1$ is symplectomorphic to $T_{p,q,r}$. During the deformation procedure, some of the critical values of $m_t$ disappear and the remaining ones correspond to the vanishing paths for the Lefschetz fibration on $\mathbb{C}^3$ defined by $\tilde{t}_{p,q,r}$. Similarly, the corresponding matching paths of the Lefschetz fibration $\hat{\pi}_T$ on $\widehat{T}_{p,q,r}$ also disappear under the deformation when $t \to 1$, and the remaining ones are the matching paths for the desired Lefschetz fibration on $T_{p,q,r}$. In this way, one gets the description of $T_{p,q,r}$ as the total space of a Lefschetz fibration $\pi_T : T_{p,q,r} \to \mathbb{C}$, whose smooth fiber is symplectomorphic to $F_{p,q}$. See Sections 2.5 and 4.3 of [60] for details.

The third step is to apply a sequence of destabilizations to the Lefschetz fibration $\pi_T$. To be self-contained, we briefly recall here the construction of a stabilization in dimension 4. Let $\pi : \overline{M} \to D^2$ be an exact symplectic Lefschetz fibration on a 4-dimensional Liouville domain $\overline{M}$ with smooth fibers $\overline{F}$ a Riemann surface with boundary. Denote by $V_1, \dots, V_n \subset \overline{F}$ the vanishing cycles of $\pi$. Given an embedded arc $\gamma \subset \overline{F}$ with $\partial\gamma \subset \partial\overline{F}$ such that $\gamma$ is an exact Lagrangian submanifold in $\overline{F}$ relative to its boundary $\partial\gamma$, one can construct a new Lefschetz fibration $\pi^s : \overline{M} \to D^2$, called the *stabilization* of $\pi$, as follows:

- replace $\overline{F}$ with another Riemann surface with boundary $\overline{F}'$, which is $\overline{F}$ with a 1-handle attached along the endpoints of $\gamma$, so that the exact Lagrangian submanifold with boundary $\gamma \subset \overline{F}$ becomes a closed curve $\gamma' \subset \overline{F}'$;

- add a new critical point to the base $D^2$ of the Lefschetz fibration $\pi$ corresponding to the new vanishing cycle $\gamma'$.

Applying the stabilization construction reversely to $\pi_T$ results in a Lefschetz fibration $\pi_T^{-s} : T_{p,q,r} \to \mathbb{C}$, whose smooth fiber $F^{-s}$ is symplectomorphic to a thrice-punctured torus, which can be regarded as a plumbing of four copies of $T^*S^1$ according to a $D_4$-tree. Denote the zero sections of these cotangent bundles by $P, Q, R$ and $T$ respectively, see Figure 4.2, they form a Lagrangian skeleton of the Liouville domain $F^{-s}$.

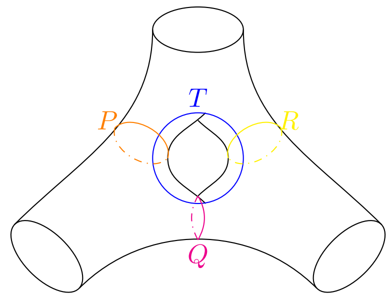


Figure 4.2: Lagrangian skeleton of the fiber of $\pi_T^{-s}$

Finally, the Milnor fiber $M_{p,q,r}$ is defined by adding a quadratic term $w^2$ to the defining equation of $T_{p,q,r}$, so obtaining a Lefschetz fibration on $M_{p,q,r}$ from the Lefschetz fibration $\pi_T^{-s}$ on $T_{p,q,r}$ can be realized as the case $r = 2$ of the situation described in the second step above. In fact, this is much simpler as the additional term $w^2$ doesn't require a Morsification [1], see Section 2.5 of [60] and also Section 6 of [92]. We denote the resulting Lefschetz fibration by

$$\pi_{p,q,r} : M_{p,q,r} \to \mathbb{C}. \tag{4.4}$$

It is clear from its definition that a smooth fiber of $\pi_{p,q,r}$ is symplectomorphic to a 4-dimensional $D_4$ Milnor fiber. By abuse of notations, the compact cores of these $D_4$-plumbings of $T^*S^2$'s will still be denoted by $P, Q, R$ and $T$.

After some mutations, the vanishing cycles of $\pi_{p,q,r}$ are given by

$$T, \tau_P^2\tau_Q^2\tau_R^2(T), \tau_P\tau_Q\tau_R(T) \tag{4.5}$$

together with $p$ copies of $P$, $q$ copies of $Q$ and $r$ copies of $R$, where $\tau_V$ denotes the Dehn twist along the vanishing cycle $V$.

[1] Alternatively, one can try to make the fibration $\pi_T^{-s}$ explicit as a polynomial on $\mathbb{C}^3$, then the Lefschetz fibration $\pi_{p,q,r}$ is the restriction of $\pi_T^{-s} + w^2$ on $\mathbb{C}^4$. This has been carried out in Section 4.1 of [22] in the special case when $p = q = 1$ and $r = 0$.

## 4.2 Casals-Murphy recipe

We recall how to obtain the Legendrian front description of a Weinstein manifold $M$ starting from a symplectic Lefschetz fibration on it. This is written down systematically by Casals-Murphy in [22].

Let $\pi : M \to \mathbb{C}$ be a Lefschetz fibration with smooth fiber $F_T$, which is a plumbing of $T^*S^{n-1}$'s according to some tree $T$. Given these data, Casals and Murphy suggest in [22] the following procedure to obtain a Legendrian handle body decomposition of $M$.

- Draw $r$ $(n-1)$-handles which correspond to the zero sections $L_1, \dots, L_r$ of $T^*S^{n-1}$ in the plumbing $F_T$.
- Find a Lefschetz fibration $\pi_F : F_T \to \mathbb{C}$ so that the Lagrangian spheres $\{L_i\}$ appear as matching cycles of $\pi_F$ with matching paths $\gamma_1, \dots, \gamma_r \subset \mathbb{C}$.
- For any vanishing cycle $V_j \subset F_T$ of $\pi$, draw the embedded path $\beta_j \subset \mathbb{C}$ under the projection of $\pi_F$.
- Express each matching path $\beta_j$ of $V_j$ as a word in half-twists along the paths in $\{\gamma_i\}$. The vanishing cycles $\{V_j\}$ are thus expressed in terms of words in Dehn twists along the Lagrangian spheres in $\{L_i\}$ of some $L_k$.
- Using handle slides, one is able to draw the front projections of their Legendrian lifts $\{\Lambda_j\}$ in the contact boundary $\partial(F_T \times D^2)$.
- The above step produces a Legendrian link $\Lambda = \bigcup_j \Lambda_j \subset \partial(F_T \times D^2)$ going through the $(n-1)$-handles. We then push each component $\Lambda_j$ of $\Lambda$ in the Reeb direction of $\partial(F_T \times D^2)$ by height $j$.

- Simplify the Legendrian front projection of $\Lambda$ using Reidemeister moves and handle cancellations.

Casals-Murphy recipe is extremely useful in obtaining Legendrian frontal descriptions of Weinstein manifolds $M$ with $\dim_{\mathbb{R}}(M) \geqslant 6$, since the existence of a Lefschetz fibration on $M$ is proved by Giroux-Pardon in [51]. In the special case when $M$ is obtained by stabilizing a 4-dimensional Milnor fiber $T$ by adding quadratic terms to its defining equation, the Legendrian surgery picture of $M$ is realized locally as an $S^{n-2}$ spin of that of $T$.

**Remark 4.2.1.** *By definition, there is an obvious Lefschetz fibration $\pi : M_{p,q,r} \to \mathbb{C}$ given by projecting to the $w$ coordinate plane, whose fiber is symplectomorphic to $T_{p,q,r}$. However, $T_{p,q,r}$ is not a plumbing of $T^*S^2$'s. In fact, it is proved by Keating in [60] that the compact Fukaya category $\mathcal{F}(T_{p,q,r})$ is not split-generated by vanishing cycles over any field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) \neq 2$. This explains why we choose to apply Casals-Murphy recipe to the destabilized Lefschetz fibration $\pi_{p,q,r}$ in Section 4.3 to get the Legendrian front associated to $M_{p,q,r}$.*

## 4.3 Legendrian surgery presentation of $M_{p,q,r}$

We now apply Casals-Murphy recipe to the Lefschetz fibration $\pi_{p,q,r} : M_{p,q,r} \to \mathbb{C}$ described in Section 4.1. Label the vanishing cycles of $\pi_{p,q,r}$ by

$$\begin{aligned} &V_{-1} = \tau_P^2\tau_Q^2\tau_R^2(T), V_0 = \tau_P\tau_Q\tau_R(T), V_1 = \cdots = V_p = P, \\ &V_{p+1} = \cdots = V_{p+q} = Q, V_{p+q+1} = \cdots = V_{p+q+r} = R, V_{p+q+r+1} = T. \end{aligned} \tag{4.6}$$

This enables us to draw the Legendrian frontal presentation of $M_{p,q,r}$ based on the data given by the Lefschetz fibration $\pi_{p,q,r}$. The picture consists of four 2-handles labelled by $P, Q, R$ and $T$ and $(p+q+r+3)$ 3-handles corresponding to the vanishing cycles $V_{-1}, \ldots, V_{p+q+r+1}$. The two non-trivial 3-handles corresponding to $V_{-1}$ and $V_0$ are depicted in Figures 4.3 and 4.4, where the thick dots in the Figure represent cone singularities. As Legendrian surfaces, they are denoted respectively by $\Lambda_A$ and $\Lambda_B$. All the other Legendrian attaching spheres $\Lambda_{P_i}$, $\Lambda_{Q_j}$, $\Lambda_{P_k}$ and $\Lambda_T$ are just parallel strands which go through a single handle $P, Q, R$ and $T$ respectively. Their interactions with the Legendrian attaching spheres $\Lambda_A$ and $\Lambda_B$ are illustrated in the Figures 4.3 and 4.4, where each set of Legendrian spheres $\{\Lambda_{P_i}\}$, $\{\Lambda_{Q_j}\}$, $\{\Lambda_{R_k}\}$ is represented by a single Legendrian sphere $\Lambda_P$, $\Lambda_Q$ and $\Lambda_R$. Note that what we have drawn in these two figures should be understood as Legendrian surfaces obtained by a number local of $S^1$-symmetric rotations of the arcs representing critical handles, see Section 2.4.2 of [22] for the detailed conventions.

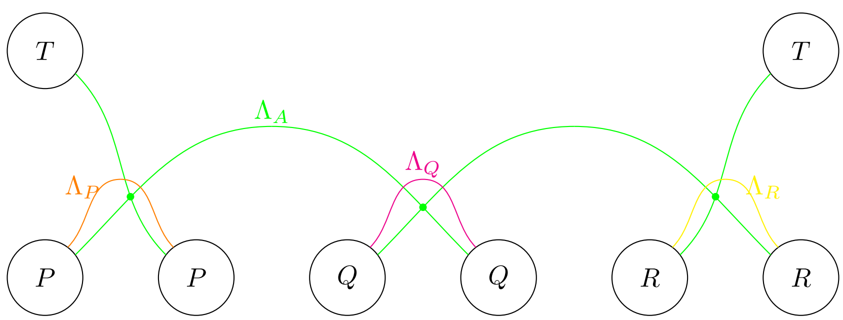


Figure 4.3: Front projection of the components $\Lambda_A$, $\Lambda_P$, $\Lambda_Q$ and $\Lambda_R$

**Proposition 4.3.1.** *The Weinstein 6-manifold $M_{p,q,r}$ is obtained by attaching Weinstein 3-handles to $D^6$ along the link of 2-dimensional Legendrian unknots*

$$\Lambda_{p,q,r} = \Lambda_A \cup \Lambda_B \cup \bigcup_{i=1}^{p-1} \Lambda_{P_i} \cup \bigcup_{j=1}^{q-1} \Lambda_{Q_j} \cup \bigcup_{k=1}^{r-1} \Lambda_{R_k}. \tag{4.7}$$

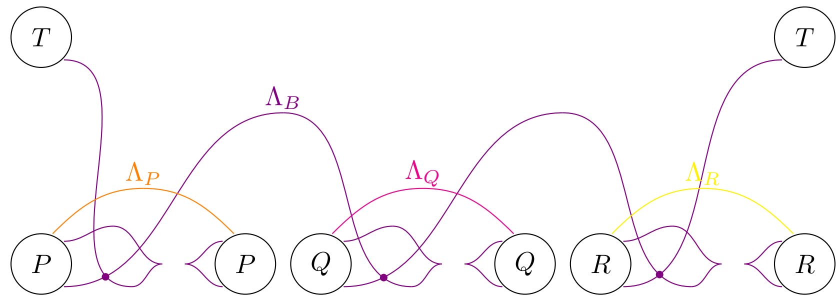


Figure 4.4: Front projection of the components $\Lambda_B$, $\Lambda_P$, $\Lambda_Q$ and $\Lambda_R$

*In particular, when $p = q = r = 2$, $\Lambda_{2,2,2}$ is Legendrian isotopic to a link of Legendrian surfaces whose front projection is depicted in Figure 4.5. In general, one can obtain the Legendrian front of $\Lambda_{p,q,r}$ by replacing the component $\Lambda_P$ (resp. $\Lambda_Q$ and $\Lambda_R$) in Figure 4.5 by an $A_{p-1}$ (resp. $A_{q-1}$ and $A_{r-1}$) chain of standard unknots which are parallel to each other. Moreover, $\Lambda_{P_1}$, $\Lambda_{Q_1}$ and $\Lambda_{R_1}$ are the only Legendrian spheres in the sets $\{\Lambda_{P_i}\}$, $\{\Lambda_{Q_j}\}$ and $\{\Lambda_{R_k}\}$ whose fronts have non-trivial intersections with the fronts of $\Lambda_A$ and $\Lambda_B$.*

*Proof.* We apply handle cancellation to the Legendrian surgery diagram associated to the Lefschetz fibration $\pi_{p,q,r} : M_{p,q,r} \to \mathbb{C}$. The vanishing cycles

$$V_1, V_{p+1}, V_{p+q+1}, V_{p+q+r+1} \tag{4.8}$$

of $\pi_{p,q,r}$ are all $(-1)$-Legendrian spheres in the vertical boundary $\partial^{\mathrm{v}}\overline{M}_{p,q,r}$ which intersect the belt spheres of the Weinstein 2-handles labelled by $P, Q, R$ and $T$, where by $(-1)$-Legendrian spheres we mean the Legendrian spheres in the boundary of the pre-surgery Weinstein domain $\overline{F}_{D_4} \times D^2$ along which the Weinstein 3-handles are attached. In particular, each of them can be cancelled with the corresponding subcritical handle, so that we obtain a Legendrian surgery diagram without 2-handles.

To further simplify the Legendrian front, we are going to use the higher dimen-

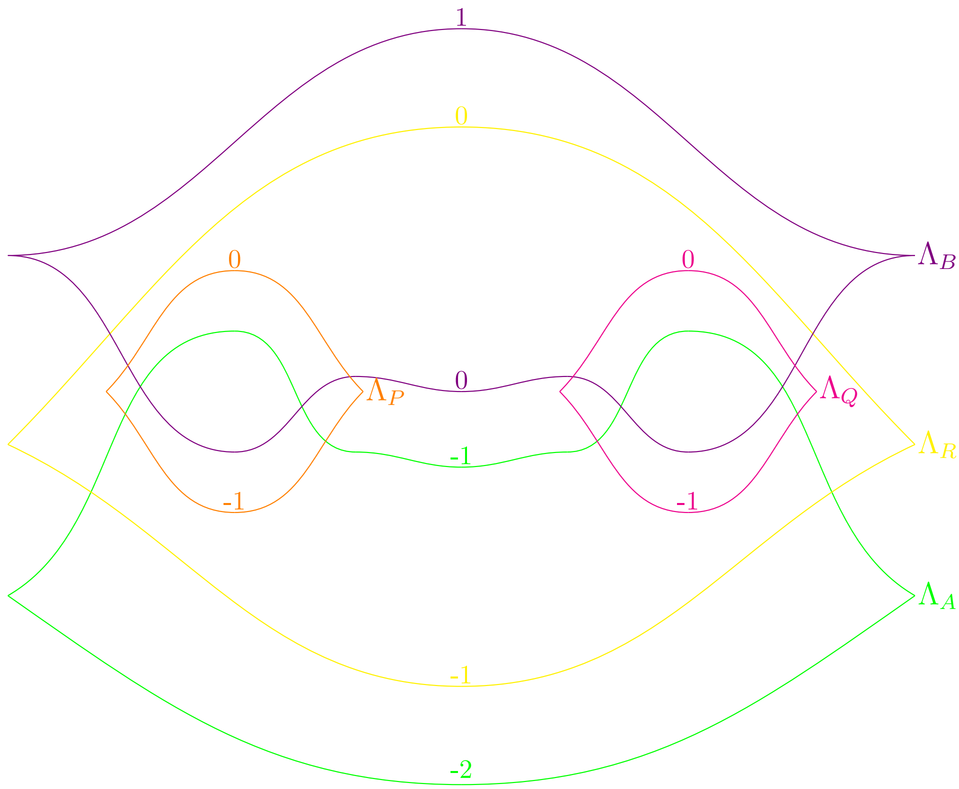


Figure 4.5: Legendrian front of $\Lambda_{2,2,2}$ after handle cancellation, where the numbers above the sheets are values taken by a Maslov potential $\mu_{2,2,2} : \Lambda_{2,2,2} \to \mathbb{Z}$

sional Reidemeister moves introduced in Section 2.4.2 of [22]. For the component $\Lambda_B$ as pictured in Figure 4.4, after cancelling all the 2-handles with $V_1$, $V_{p+1}$, $V_{p+q+1}$ and $V_{p+q+r+1}$, we get a Legendrian knot as pictured in the upper half of Figure 4.6. After a sequence of Reidemeister I and Reidemeister II moves, one can simplify it to a Legendrian 2-sphere with three cone singularities, as is shown in the lower left of Figure 4.6. This Legendrian 2-sphere can be seen to be Legendrian isotopic to the standard unknot by applying the Legendrian isotopy pictured in Figure 4.7. Note that the move in Figure 4.7 is a Legendrian isotopy since it is the $S^1$-rotation of the Reidemeister I move with respect to the axis which passes through the crossing point.

Similarly, the Legendrian knot $\Lambda_A$ can be seen to be Hamiltonian isotopic to the standard unknot after cancelling all the 2-handles. The fact that after handle cancellation, the components $\Lambda_{P_i}$, $\Lambda_{Q_j}$ and $\Lambda_{R_k}$ are isotopic to the standard unknots is obvious.

One dimensional lower, the Legendrian front of the affine surface $T_{p,q,r}$ is a link of unknots $K_{p,q,r} \subset (\mathbb{R}^3, \xi_{std})$. The linking numbers between the components

$$K_A, K_B, K_{P_i}, K_{Q_j}, K_{R_k} \subset K_{p,q,r} \tag{4.9}$$

are determined by the intersection numbers of their Lagrangian fillings, which, after capping off using the core discs of the 2-handles attached along $K_{p,q,r}$, becomes a basis of vanishing cycles of the Milnor fiber $T_{p,q,r}$. For example, the Lagrangian fillings of $K_A$ and $K_B$ are Lagrangian discs intersecting transversely at two points, which implies that the corresponding Lagrangian cocore discs $L_A$ and $L_B$ of the 2-handles attached to $K_A$ and $K_B$ also have transversal intersections at two points. Since each of the intersection points in $L_A \cap L_B$ gives rise to two Reeb chords, one from $K_A$ to $K_B$, and the other one from $K_B$ to $K_A$, we see that the front projections of $K_A$

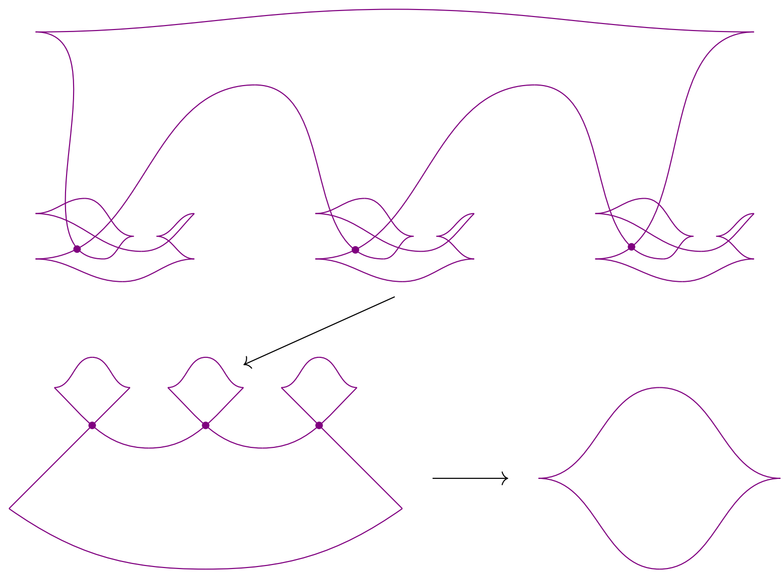

Figure 4.6: Front projection of $\Lambda_B$ after handle cancellation, and its simplifications

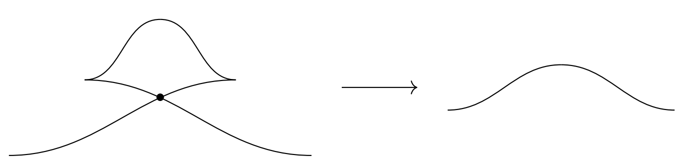

Figure 4.7: Symmetric rotation of the Reidemeister I move

and $K_B$ intersect at 4 points. Since the Legendrian front of $\Lambda_{p,q,r}$ is given locally by an $S^1$-symmetric rotation of $K_{p,q,r}$, the intersections of the different components of $\Lambda_{p,q,r}$ under the front projection is determined by that of $K_{p,q,r}$. In particular, the fronts of $\Lambda_A$ and $\Lambda_B$ intersect along two circles. $\square$

# Chapter 5

# Cellular dg algebra

This chapter is an exposition of the paper [85] by Rutherford-Sullivan, which gives a combinatorial model of the Chekanov-Eliashberg dg algebra $CE^*(\Lambda)$ in the case when $\Lambda \subset (\mathbb{R}^5, \xi_{std})$ is a Legendrian surface. We will be mainly focusing on the definitions and results that are relevant to our computations in Chapter 6. In this chapter, we fix the coefficient field $\mathbb{K}$ to be $\mathbb{Z}/2$.

## 5.1 The definition

We recall here the definition of the cellular dg algebra associated to a closed Legendrian surface $\Lambda \subset (S^5, \xi_{std})$, with the additional assumption that there is no swallowtail singularity in the front projection of $\Lambda$.

Let $S$ be a surface. A *polygonal decomposition* of $S$ is a decomposition of $S$ into CW complexes

$$S = \bigsqcup_{i=0}^{2} \bigsqcup_{\alpha} e_{\alpha}^{i} \tag{5.1}$$

equipped with characteristic maps $c_\alpha^i : D^i \to S$ which satisfy the following two properties:

- $c_\alpha^1$ are smooth for all $\alpha$.
- For any 2-cell $e_\alpha^2$, pre-images of 0-cells divide $\partial D^2$ into intervals that are mapped homeomorphically to 1-cells by $c_\alpha^2$.

Denote by $J^1(S) := T^*S \times \mathbb{R}$ the 1-jet space. Let $\Lambda \subset J^1(S)$ be a Legendrian surface. Then there are two natural projections, namely the front projection

$$p_{x,z} : J^1(S) \to S \times \mathbb{R} \tag{5.2}$$

and the base projection

$$p_x : J^1(S) \to S, \tag{5.3}$$

where $x = (x_1, x_2)$ denotes the local coordinates on $S$ and $z$ is coordinate in the Reeb direction $\mathbb{R}$.

Suppose that $\Lambda$ has generic front projection, denote by $\Sigma \subset S$ the image of the singular set of $\Lambda$ under the base projection $p_x$, then it decomposes as $\Sigma = \Sigma_1 \sqcup \Sigma_2$, where $\Sigma_i$ denotes the base projection of the set of codimension $i$ singularities. A polygonal decomposition of $S$ is $\Lambda$-*compatible* if $\Sigma$ is contained in the 1-skeleton $\bigcup_\alpha e_\alpha^1$ of the polygonal decomposition. It then follows that $\Sigma_2$ is contained in the 0-skeleton $\bigsqcup_\alpha e_\alpha^0$.

We now proceed to define the cellular dg algebra

$$(\mathcal{C}(\Lambda), d_{\mathcal{C}}) \tag{5.4}$$

associated to the Legendrian surface $\Lambda \subset J^1(S)$.

We first describe the set of generators of $\mathcal{C}(\Lambda)$. Given a $\Lambda$-compatible polygonal decomposition of $S$, which has $e^i_\alpha$ as one of its cells, the connected components $\Lambda \cap p_x^{-1}(e^i_\alpha)$ which are not contained in any cusp edge will be referred to as *sheets* of $\Lambda$ above $e^i_\alpha$. Denote by $\Lambda(e^i_\alpha)$ the set of sheets of $\Lambda$ above $e^i_\alpha$, using their $z$-coordinates we can equip $\Lambda(e^i_\alpha)$ with a partial ordering $\prec$. More precisely, for $S_1, S_2 \in \Lambda(e^i_\alpha)$, $S_1 \prec S_2$ if $z(S_1) > z(S_2)$ above $e^i_\alpha$. Note that two sheets are incomparable if and only if they meet in a crossing arc above $e^i_\alpha$ in the front projection of $\Lambda$.

For each cell $e^i_\alpha$ in the $\Lambda$-compatible polygonal decomposition, we associate one generator of $\mathcal{C}(\Lambda)$ for each pair of sheets $S_m, S_n \in \Lambda(e^i_\alpha)$ with $S_m \prec S_n$. These generators will be denoted by $a^{m,n}_\alpha$, $b^{m,n}_\alpha$ and $c^{m,n}_\alpha$ when the corresponding cells are 0-dimensional, 1-dimensional and 2-dimensional respectively. As a graded algebra, $\mathcal{C}(\Lambda)$ is freely generated by these generators.

We shall be interested in the case when $\Lambda = \bigsqcup_v \Lambda_v$ is a link of Legendrian surfaces. In this case, $\mathcal{C}(\Lambda)$ carries the structure of a $\Bbbk$-bimodule, with $\Bbbk = \bigoplus_v \mathbb{K}e_v$ being a semisimple ring, which we describe as follows. First, consider the vector spaces $\mathbb{K}\langle a^{m,n}_\alpha, b^{m,n}_\alpha, c^{m,n}_\alpha\rangle$ spanned by the generators in $A_\alpha, B_\alpha$ and $C_\alpha$, we can endow it with a $\Bbbk$-bimodule structure by declaring

$$e_w\{a^{m,n}_\alpha, b^{m,n}_\alpha, c^{m,n}_\alpha\}e_v \tag{5.5}$$

to be the set of generators associated to the pair of sheets $(S_m, S_n)$ with $S_m \subset \Lambda_v$ and $S_n \subset \Lambda_w$. As a $\Bbbk$-bimodule, $\mathcal{C}(\Lambda)$ is the tensor algebra over $\Bbbk$ defined by

$$\mathcal{C}(\Lambda) := \bigoplus_{i=0}^{\infty} \mathbb{K}\langle a^{m,n}_\alpha, b^{m,n}_\alpha, c^{m,n}_\alpha\rangle^{\otimes_{\Bbbk} i}. \tag{5.6}$$

We shall assume from now on that the Maslov class of $\Lambda$ vanishes. Since we will deal only with the case when $\Lambda$ is a disjoint union of Legendrian spheres, this assumption is automatically satisfied for all the examples studied in this thesis. This allows us to endow $\mathcal{C}(\Lambda)$ with a $\mathbb{Z}$-grading. Fix a *Maslov potential*

$$\mu : \Lambda \to \mathbb{Z}, \tag{5.7}$$

which is a locally constant function whose value increases by 1 when passing from the lower sheet to the upper sheet at a cusp edge. Each of the generators $a_\alpha^{m,n}, b_\alpha^{m,n}, c_\alpha^{m,n}$ is homogeneous in $\mathcal{C}(\Lambda)$, and their degrees are specified as follows:

$$|a_\alpha^{m,n}| = \mu(S_n) - \mu(S_m) + 1, |b_\alpha^{m,n}| = \mu(S_n) - \mu(S_m), |c_\alpha^{m,n}| = \mu(S_n) - \mu(S_m) - 1. \tag{5.8}$$

Note that our convention here is different from that of [85] as we shall use a cohomological grading on $\mathcal{C}(\Lambda)$.

In order to define the differential $d_{\mathcal{C}}$, we set $d_{\mathcal{C}}(1) = 0$, and specify the action of $d_{\mathcal{C}}$ on the generators $a_\alpha^{m,n}$, $b_\alpha^{m,n}$ and $c_\alpha^{m,n}$ of $\mathcal{C}(\Lambda)$ separately.

Consider a 0-cell $e_\alpha^0$. By extending the partial ordering $\prec$ on $\Lambda(e_\alpha^0)$ we can define a total ordering on the set of sheets above $e_\alpha^0$, which can be equivalently described by a bijection

$$\rho : \{1, \dots, r\} \to I_\alpha, \tag{5.9}$$

where $I_\alpha$ is the index set recording the subscripts of the sheets in $\Lambda(e_\alpha^0)$. This total ordering on $\Lambda(e_\alpha^0)$ enables us to assemble the generators $a_\alpha^{m,n}$ in a strictly upper triangular matrix $A_\alpha$, with its $(i,j)$-th entry given by $a_\alpha^{\rho(i),\rho(j)}$ if $S_{\rho(i)} \prec S_{\rho(j)}$, and 0 otherwise. $d_{\mathcal{C}}$ on the generators $a_\alpha^{m,n}$ is then determined by the matrix equation

$$d_{\mathcal{C}} A_\alpha = A_\alpha^2, \tag{5.10}$$

where on the left-hand side $d_{\mathcal{C}}$ is applied entrywisely. It is proved in [85] that $d_{\mathcal{C}} a_\alpha^{m,n}$ is independent of the choice of the total ordering on $\Lambda(e_\alpha^0)$ which extends $\prec$.

In the case of a 1-cell $e^1_\alpha$, we can again enhance the partially ordered set $(\Lambda(e^1_\alpha), \prec)$ to get a totally ordered set by specifying a bijection $\rho$ as above. This then gives us a strictly upper triangular $r\times r$ matrix $B_\alpha$ whose $(i,j)$-th entry equals $b^{\rho(i),\rho(j)}_\alpha$ if $S_{\rho(i)} \prec S_{\rho(j)}$, and $b^{\rho(i),\rho(j)}_\alpha = 0$ otherwise. Since the structure of a polygonal decomposition of $S$ includes as its data a set of characteristic maps $c^i_\alpha : D^i \to S$, we are allowed to distinguish between the initial and terminal 0-cells $e^0_{\alpha,+}$ and $e^0_{\alpha,-}$. Notice however that a 1-cell can have identical initial and terminal points. For the 0-cells $e^0_{\alpha,+}$ and $e^0_{\alpha,-}$, we can associate to them two $r \times r$ matrices $A_{\alpha,+}$ and $A_{\alpha,-}$ as follows.

Each sheet above the 0-cell $e^0_{\alpha,+}$ belongs to the closure of a unique sheet in $\Lambda(e^1_\alpha)$. Under the bijection $\rho$, this induces an order-preserving injective map

$$\iota : \Lambda(e^0_{\alpha,+}) \hookrightarrow \{1, \ldots, r\}. \tag{5.11}$$

Those sheets of $e^1_\alpha$ not in the image of $\iota$ meet in pairs at cusp points above $e^0_{\alpha,+}$. The $(i,j)$-th entry of $A_{\alpha,+}$ is defined to be $a^{m,n}_{\alpha,+}$ if $S_m \prec S_n$ and $\iota(m) = i, \iota(n) = j$. The $(k, k+1)$-th entry of $A_{\alpha,+}$ is 1 if the sheets numbered $k$ and $k+1$ of $\Lambda(e^1_\alpha)$ under $\rho$ meet at a cusp singularity above $e^0_{\alpha,+}$. All the other entries of $A_{\alpha,+}$ are set to be 0. Alternatively, one can use the total ordering specified by $\iota$ to form a matrix out of the generators $a^{m,n}_{\alpha,+}$, and then insert $2 \times 2$ blocks $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ along the diagonal for each pair of sheets of $\Lambda(e^1_\alpha)$ that meet at a cusp edge above $e^0_{\alpha,+}$. The definition of $A_{\alpha,-}$ is completely identical.

With these matrices at hand, $d_{\mathcal{C}} b^{m,n}_\alpha$ can be defined by the following matrix equation:

$$d_{\mathcal{C}} B_\alpha = A_{\alpha,+}(E + B_\alpha) + (E + B_\alpha)A_{\alpha,-}, \tag{5.12}$$

with $E$ being the identity matrix. Again, $d_{\mathcal{C}} b^{m,n}_\alpha$ is independent of the choice of the total ordering on $\Lambda(e^1_\alpha)$.

When it comes to a 2-cell $e^2_\alpha$, the partial ordering on $\Lambda(e^2_\alpha)$ is already a total ordering, so we can label the sheets in $\Lambda(e^2_\alpha)$ directly by $S_1, \ldots, S_r$ so that

$$z(S_1) > \cdots > z(S_r). \tag{5.13}$$

The sheets above the edges and vertices in $\partial e^2_\alpha$ are therefore naturally identified with subsets of $\{1, \ldots, r\}$. For each such edge or vertex we can define a strictly upper triangular $r \times r$ matrix by using the corresponding generators $b^{m,n}_\alpha$ or $a^{m,n}_\alpha$ and $2 \times 2$ blocks

$$O := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{(for an edge)}, N := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{(for a vertex)} \tag{5.14}$$

inserted in the diagonal whenever $S_k, S_{k+1}$ meet at a cusp singularity above the edge or vertex. Just as what we have done for $A_{\alpha,+}$ and $A_{\alpha,-}$ above. Notice that in the case when $\Lambda = \bigsqcup_v \Lambda_v$ is a link of Legendrian 2-spheres and the cusp singularity formed by $S_k, S_{k+1}$ belongs to the component $\Lambda_v \subset \Lambda$, the non-zero entry in the matrix $N$ above should be replaced by the idempotent $e_v$ in the semisimple ring $\Bbbk$.

For each 2-cell $e^2_\alpha$, the characteristic map $c^2_\alpha : D^2 \to S$ determines the initial and terminal vertices $v^\alpha_-, v^\alpha_+ \in \partial D^2$, whose associated matrices defined in the last paragraph will again be denoted by $A_{\alpha,-}$ and $A_{\alpha,+}$. Let $\gamma_+$ and $\gamma_-$ be the arcs in $\partial D^2$ that go counterclockwisely and clockwisely from $v_-$ to $v_+$ respectively. Note that these paths can be constant or the entire circle. Consider the image of $c^2_\alpha \circ \gamma_+$, it contains a set of successive 1-cells $e^1_{\alpha,1}, \ldots, e^1_{\alpha,n_+}$, whose associated matrices defined in the last paragraph will be denoted by $B_{\alpha,1}, \ldots, B_{\alpha,n_+}$. Similarly, for the path $\gamma_-$ we get another set of 1-cells $e^1_{\alpha,n_++1}, \ldots, e^1_{\alpha,n_++n_-}$ with associated matrices $B_{\alpha,n_++1}, \ldots, B_{\alpha,n_++n_-}$. Finally, similar to the cases of 0-cells and 1-cells, we can form the (strictly upper triangular) matrix $C_\alpha$ using the generators $c^{m,n}_\alpha$ corresponding to the 2-cell $e^2_\alpha$. Now the differential $d_{\mathcal{C}}$ on $c^{m,n}_\alpha$ is defined via the matrix equation

$$\begin{aligned} d_{\mathcal{C}} C_\alpha = A_{\alpha,+} C_\alpha + C_\alpha A_{\alpha,-} + (E + B_{\alpha,n_+})^{\varepsilon_{n_+}} \dots (E + B_{\alpha,1})^{\varepsilon_1} \\ + (E + B_{\alpha,n_+ + n_-})^{\varepsilon_{n_+ + n_-}} \dots (E + B_{\alpha,n_+ + 1})^{\varepsilon_{n_+ + 1}}, \end{aligned} \tag{5.15}$$

where $\varepsilon_i = 1$ if the orientation on the 1-cell $e^+_{\alpha,i}$ as an edge of $e^2_\alpha$ coincides with the orientation determined by the characteristic map of the 1-cell, otherwise $\varepsilon_i = -1$.

In all cases, it can be checked that $d^2_{\mathcal{C}} = 0$. This defines the cellular dg algebra $(\mathcal{C}(\Lambda), d_{\mathcal{C}})$ when there is no swallowtail points in the front projection of $\Lambda$. When $\Lambda = \bigsqcup_v \Lambda_v$ is a link of Legendrian surfaces, one can check that the differential $d_{\mathcal{C}}$ defined above is compatible with the $\mathbb{k}$-bimodule structure (5.5) on $\mathcal{C}(\Lambda)$, which shows that $\mathcal{C}(\Lambda)$ is a dg algebra over $\mathbb{k}$. The definition of $\mathcal{C}(\Lambda)$ can be extended to the case when swallowtail singularities present in the front projection of $\Lambda$, with some modifications to the matrices $A_\alpha$, $B_\beta$ and $C_\gamma$. Since this will not be used for later computations, its definition will not be recalled here. See Section 3.11 of [85] for details.

Up to quasi-isomorphism, the dg algebra $\mathcal{C}(\Lambda)$ is independent of the choice of the polygonal decomposition. Together with the quasi-isomorphism (6.154) established in [86], we see that the quasi-isomorphism type of $\mathcal{C}(\Lambda)$ defines an invariant of $\Lambda$ under Legendrian isotopy.

## 5.2 Non-genericity

For computational convenience, we will allow certain type of non-generic Legendrian fronts, namely when there are multiple crossings or cusp edges above a 1-cell in the cellular decomposition. In these cases, we can modify slightly our original definitions of the cellular dg algebra $(\mathcal{C}(\Lambda), d_{\mathcal{C}})$ to get a (usually simpler) dg algebra $\left(\mathcal{C}^\ell(\Lambda), d^\ell_{\mathcal{C}}\right)$, whose definition we will recall below.

We deal first with the case when multiple crossing arcs appear above some subset of $\Sigma_1 \subset S$. To be precise, let $\Lambda \subset J^1(S)$ be a Legendrian surface. Consider a polygonal decomposition of $p_x(\Lambda)$ which is $\Lambda$-compatible except near a 2-sided simple closed curve $\ell \subset S$. Suppose that in a neighborhood $\overline{U} \cong S^1 \times [0,1]$ of $\ell := S^1 \times \{\frac{1}{2}\}$, there are several crossing arcs of $\Lambda$ which project to small shifts of $\ell$ in the normal direction, and no other crossings or cusp edges. In this case, we can assume that all the crossing arcs near $\ell$ project precisely to $\ell$ and define the generators in the dg algebra $\mathcal{C}^\ell(\Lambda)$ associated to this incompatible cellular decomposition as follows. See Figure 5.1 for an illustration.

More precisely, label the sheets of $\Lambda$ above one side of the neighborhood $\overline{U}$ as $S_1, \ldots, S_r$, so that $z(S_1) > \cdots > z(S_r)$ above that side of $\overline{U}$. The key point is that there exists a permutation $\sigma$ on the set $\{1, \ldots, r\}$ such that the sheet $S_i$ appears as the sheet $S_{\sigma(i)}$ on the other side of the small neighbourhood $\overline{U}$, so that all the crossings of the sheets of $\Lambda$ happened in $\overline{U}$ (or equivalently, one can treat them as crossings over $\ell \subset \overline{U}$ ) are recorded by this permutation.

To each 1-cell $e^1_\beta$ (resp. 0-cell $e^0_\alpha$) of $\ell$, assign generators $b^{m,n}_\beta$ (resp. $a^{m,n}_\alpha$) for all $m < n$ with $\sigma(m) < \sigma(n)$, so that there are multiple zeros in the corresponding matrices $B_\beta$ (resp. $A_\alpha$) of generators, since sheets which cross with each other in $\overline{U}$ will then satisfy $\sigma(m) > \sigma(n)$. The differential $d^\ell_{\mathcal{C}}$ is defined by the same formulas (5.10), (5.12) and (5.15) as in the case of a usual cellular dg algebra $\mathcal{C}(\Lambda)$. Note that if $B_\beta$ is the matrix of generators associated to the sheets labelled by $S_1, \ldots, S_r$, then the corresponding matrix of generators associated to the sheets labelled by $S_{\sigma^{-1}(1)}, \ldots, S_{\sigma^{-1}(r)}$ is $Q_\sigma^{-1} B_\beta Q_\sigma$, with $Q_\sigma = \sum_{m=1}^r \Delta_{\sigma(m),m}$, and $\sigma$ being a composition of transpositions on $\{1, \ldots, r\}$.


Figure 5.1: Local picture of a $\Lambda$-compatible polygonal decomposition associated to a non-generic Legendrian front of $\Lambda$, which projects several crossings/cusp edges to the unique 1-cell $\ell$ (left), and the neighbourhood $\overline{U}$ of a $\Lambda$-compatible polygonal decomposition associated to a generic Legendrian front of $\Lambda$, where different crossings/cusp edges are projected to multiple 1-cells, the dg algebra $(\mathcal{C}^{\ell}(\Lambda), d_{\mathcal{C}}^{\ell})$ is defined by assembling these 1-cells to $\ell$ by making the front of $\Lambda$ non-generic (right)

When several cusp edges appear above the curve $\ell$, the situation is much simpler. For each 0-cell (resp. 1-cell) of $\ell$, it follows from (5.14) that we only need to insert multiple copies of $2 \times 2$ nilpotent blocks $N$ (resp. zero blocks $O$) in the definition of the matrices $A_\alpha$ (resp. $B_\beta$). For example, let $\Lambda \subset (\mathbb{R}^5, \xi_{std})$ be a Legendrian surface so that locally its base projection $p_x(\Lambda)$ is depicted on the left-hand side of Figure 5.2. Suppose that the Legendrian front of $\Lambda$ above the 0-cell $e_\alpha^0$ (resp. 1-cell $e_\beta^1$) consists only of cusp edges formed by couples of sheets $(S_1, S_2), \cdots, (S_{2r-1}, S_{2r})$, and the ordering of these sheets is chosen so that $z(S_1) > \cdots > z(S_{2r})$ locally in a small neighborhood on the right-hand side of the solid arc labelled $e_\beta^1$. In this case, one can take

$$A_\alpha = \sum_{i=1}^{r} \Delta_{2i-1,2i}, B_\beta = 0. \tag{5.16}$$

Again, the relevant differentials are defined by the same formulas as in the front generic case. This completes the definition of $(\mathcal{C}^{\ell}(\Lambda), d_{\mathcal{C}}^{\ell})$.

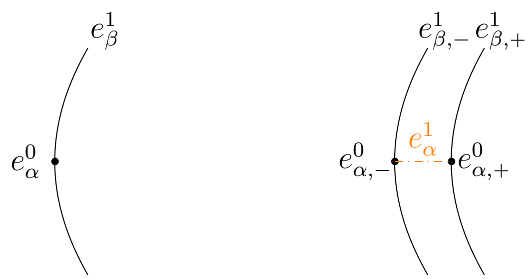


Figure 5.2: The base projections $p_x$ of a Legendrian surface $\Lambda$ near its cusp edges, where the images of $\Lambda$ under $p_x$ appear on the right-hand side of the leftmost solid arcs in both of the figures. On the left hand side, all the cusp edges in the front of $\Lambda$ formed by the sheets $(S_1, S_2), \cdots, (S_{2r-1}, S_{2r})$ are projected to the same solid arc labelled as the 1-cell $e^1_\beta$. On the right hand side, we are in the special case when $r = 2$, and a small Legendrian isotopy of $\Lambda$ has been chosen so that the base projections of the cusp edges formed by $(S_3, S_4)$ and $(S_1, S_2)$ are projected to different solid arcs labelled respectively by $e^1_{\beta,-}$ and $e^1_{\beta,+}$.

The following proposition can be proved by applying small isotopies to $\Lambda$ to make it front generic, and then simplifying $\mathcal{C}(\Lambda)$ using stable tame isomorphisms. The latter step can be achieved by applying Lemma 5.3.1, which will be recalled in the next section, to the original definition of $\mathcal{C}(\Lambda)$ recorded in Section 5.1.

**Proposition 5.2.1** (Proposition 5.5 of [85]). *The dg algebra $\mathcal{C}^{\ell}(\Lambda)$ defined above using a $\Lambda$-compatible polygonal decomposition associated to a non-generic Legendrian front of $\Lambda$ is quasi-isomorphic to the cellular dg algebra $\mathcal{C}(\Lambda)$ defined using any $\Lambda$-compatible cellular decomposition associated to a generic front projection of $\Lambda$.*

To see the how Proposition 5.2.1 can be used to simplify our computations, we consider the simplest case when the non-generic Legendrian front of $\Lambda$ above the 1-cell $e^1_\beta$ consists of two cusp edges formed by the pairs of sheets $(S_1, S_2)$ and $(S_3, S_4)$ respectively, see the left-hand side of Figure 5.2, where as before we require that $z(S_1) > z(S_2) > z(S_3) > z(S_4)$ locally on the right-hand side of $e^1_\beta$. By (5.16), we see that $A_\alpha = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix}$ in the dg algebra $\mathcal{C}^{\ell}(\Lambda)$. However, after applying a small isotopy to make $\Lambda$ front generic, the 0-cell $e^0_\alpha$ in the original cellular decomposition is replaced by two 0-cells $e^0_{\alpha,-}$ and $e^0_{\alpha,+}$ in the $\Lambda$-compatible cellular decomposition, and correspondingly the 1-cell $e^1_\beta$ is replaced by the 1-cells $e^1_{\beta,-}$ and $e^1_{\beta,+}$, see the right-hand side of Figure 5.2. More explicitly, we require that above the 1-cell $e^1_{\beta,+}$, the top two sheets meet above the cusp edge. By definition of the cellular dg algebra $\mathcal{C}(\Lambda)$ in Section 5.1, $A_{\alpha,-} = N$. To find the values of $A_{\alpha,+}$, recall from (5.11) the definition of the map $\iota : \Lambda(e^0_{\alpha,+}) \to \{1, 2, 3, 4\}$. Since in this case the sheets numbered

1 and 2 of $\Lambda(e^1_{\beta,+})$ meet at a cusp edge, we see that

$$A_{\alpha,+} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{3,4}_{\alpha,+} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.17}$$

Consider the 1-cell $e^1_\alpha$ which connects $e^0_{\alpha,-}$ to $e^0_{\alpha,+}$ on the right-hand side of Figure 5.2, it follows from (5.12) that

$$d_{\mathcal{C}} b^{3,4}_\alpha = a^{3,4}_{\alpha,+} + 1, \tag{5.18}$$

which shows the existence of a quasi-isomorphism $\mathcal{C}(\Lambda) \cong \mathcal{C}(\Lambda)/\langle a^{3,4}_{\alpha,+} + 1, b^{3,4}_\alpha \rangle$, under which $A_{\alpha,+}$ simplifies to $A_\alpha$.

In general, Proposition 5.2.1 says that by applying Lemma 5.3.1 to the formula (5.12), $A_{\alpha,+}$ in the cellular dg algebra $\mathcal{C}(\Lambda)$ can finally be simplified to the form of $A_\alpha$ in the dg algebra $\mathcal{C}^\ell(\Lambda)$, which is quasi-isomorphic to $\mathcal{C}(\Lambda)$. It is therefore more convenient to start with a non-generic front of $\Lambda$ and replace $\mathcal{C}(\Lambda)$ with its quotient dg algebra $\mathcal{C}^\ell(\Lambda)$. In this way, the formal generators in the matrix $A_{\alpha,+}$ are therefore ignored, and one can work directly with the matrix $A_\alpha$, whose entries are purely scalars. This simplification will be used frequently below, in various computations and arguments in Chapter 6.

## 5.3 Suspension of a dg algebra

Let $(\mathcal{A}, d)$ be a *based* dg algebra over $\mathbb{K}$, which means that it is freely generated over $\mathbb{K}$ by a set of homogeneous elements $a_1, \ldots, a_n$. Fix an absolute ordering $a_1 < \cdots < a_n$ on the generating set $\{a_1, \ldots, a_n\}$, there is then a natural filtration $F^\bullet$ on $\mathcal{A}$ with

$F^0\mathcal{A} = \mathbb{K}$ and $F^i\mathcal{A} \subset \mathcal{A}$ is the subalgebra generated by $a_1, \ldots, a_i$. We remark that in general, this filtration $F^\bullet$ has nothing to do with the grading on $\mathcal{A}$. Recall that the differential $d$ is said to be *triangular* if for all $1 \leqslant i \leqslant n$, $da_i \in F^{i-1}\mathcal{A}$.

A map between based dg algebras $\phi : \mathcal{A} \to \mathcal{B}$ is a *tame isomorphism* if it is the composition of a sequence of elementary automorphisms of $\mathcal{A}$ followed by an identification between the sets of generators of $\mathcal{A}$ and $\mathcal{B}$. By an *elementary tame automorphism* of $\mathcal{A}$ we mean a dg algebra automorphism which sends a fixed generator $a_i \in \mathcal{A}$ to $a_i + v$, where $v$ belongs to the dg subalgebra of $\mathcal{A}$ generated by $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$. In general, any tame isomorphism $\phi$ can be expressed as the composition of a sequence of elementary tame automorphisms of $\mathcal{A}$ followed by an identification between the sets of generators of $\mathcal{A}$ and $\mathcal{B}$.

Denote by $(S_i\mathcal{A}, d_i)$ the *degree $i$ stabilization* of $\mathcal{A}$. This is the dg algebra with generating set consisting of the original generators of $\mathcal{A}$, together with two additional elements $a, b \in S_i\mathcal{A}$ such that $|a| = |b| + 1 = i$, which satisfy

$$d_i a = b, d_i b = 0 \tag{5.19}$$

and $d_i$ coincides with $d$ when restricted to the dg subalgebra $\mathcal{A} \subset S_i\mathcal{A}$. Two dg algebras $\mathcal{A}$ and $\mathcal{B}$ are *stable tame isomorphic* if after stabilizing $\mathcal{A}$ and $\mathcal{B}$ for (possibly different) finite number of times, they become tame isomorphic to each other. We record the following simple result, whose proof dates back to the fundamental paper of Chekanov [27], which is useful in simplifying dg algebras up to stable tame isomorphism.

**Lemma 5.3.1.** *Let $(\mathcal{A}, d)$ be a based dg algebra over $\mathbb{K}$ such that $d$ is triangular with respect to the ordered generating set $\{a_1, \ldots, a_n\}$, and*

$$da_i = a_j + v, v \in F^{j-1}\mathcal{A}. \tag{5.20}$$

*Then we have a stable tame isomorphism* $(\mathcal{A}, d) \cong (\mathcal{A}/\mathcal{I}, d)$ *between based dg algebras, where* $\mathcal{I} \subset \mathcal{A}$ *is the 2-sided ideal generated by* $a_i$ *and* $da_i$*.*

The above lemma will be repeatedly applied to the cellular dg algebra $\mathcal{C}(\Lambda)$ in Chapter 6 to cancel excessive generators, and we will denote by $\mathcal{C}'(\Lambda)$ the simplified dg algebra obtained by applying all the possible cancellations to $\mathcal{C}(\Lambda)$ in the sense of Lemma 5.3.1. Note that this does not in general lead to a model of the Chekanov-Eliashberg algebra $CE^*(\Lambda_{p,q,r})$ that is minimal, especially when $CE^*(\Lambda_{p,q,r})$ is formal as a dg algebra. The Legendrian surface $\Lambda_{1,1,0}$ mentioned in Section 6.4 is such an example.

The following notion is useful in the computation of the Chekanov-Eliashberg dg algebras for Legendrian surfaces which can be realized as front spins of Legendrian knots, or more generally, Legendrian arcs. Its geometric application is discussed in Section 5.4.

**Definition 5.3.1** ([85])**.** *Let* $(\mathcal{A}, d)$ *be a dg algebra over* $\mathbb{K}$ *freely generated by the ordered set of generators* $\{a_1, \dots, a_n\}$*, and the differential* $d$ *is triangular with respect to these generators. We define the suspension* $(\mathcal{A}^s, d^s)$ *of* $(\mathcal{A}, d)$ *to be the dg algebra freely generated by* $\{a_i, \check{a}_i\}_{i=1,\dots,n}$ *such that*

- $|\check{a}_i| = |a_i| - 1$*, where the grading of* $a_i$ *in* $\mathcal{A}^s$ *is the same as its grading in* $\mathcal{A}$*.*
- *Let* $D : \mathcal{A} \to \mathcal{A}^s$ *be the derivation determined by* $D(a_i) = \check{a}_i$*, then the differential* $d^s$ *on* $\mathcal{A}^s$ *is determined by*

$$d^s(a_i) = d(a_i), d^s(\check{a}_i) = D(d(a_i)). \tag{5.21}$$

*Let* $\mathcal{O} \subset \mathcal{A}$ *be a based dg subalgebra generated by some subset of* $\{a_1, \dots, a_n\}$*. Define the suspension of* $\mathcal{A}$ *relative to* $\mathcal{O}$ *to be the dg algebra* $\mathcal{A}^s$ *as above except that we set* $\check{o} = 0$ *for all* $o \in \mathcal{O}$*.*

## 5.4 Spinning a Legendrian arc

Let $J^1(\mathbb{R})$ denote $\mathbb{R}^3$ with its standard contact structure. The corresponding front and base projections will be denoted by $p_{x_1,z}$ and $p_{x_1}$, respectively. Given a Legendrian submanifold $K \subset J^1(\mathbb{R})$ which is a union of a number of knots in $\{x_1 < 0\}$ and a number of arcs in $\{x_1 \leqslant 0\}$ whose endpoints under the front projection lie on the $z$-axis. In addition, we shall require that when we reflect $p_{x_1,z}(K) \subset \mathbb{R}^2$ with respect to the $z$-axis, the image should be the front projection of a Legendrian link. This can actually be achieved for any Legendrian embedding $K \subset \mathbb{R}^3$ whose connected components are knots and arcs by a suitable Legendrian isotopy. By rotating along the $z$-axis, such a $K \subset J^1(\mathbb{R})$ gives rise to a Legendrian surface $\Lambda_K \subset J^1(\mathbb{R}^2)$, whose front $p_{x,z}(\Lambda_K) \subset \mathbb{R}^3$ is also the rotation of $p_{x_1,z}(K) \subset \mathbb{R}^2$ along the same axis. We call $\Lambda_K$ the *front spin* of $K$. It is easy to see that the surface $\Lambda_K$ is a disjoint union of Legendrian tori and Legendrian spheres. Note that when two endpoints of a Legendrian arc coincide with each other on the $z$-axis under the front projection, then its front spin will have a cone singularity. In this case, although $\Lambda_K$ is not front generic, its cellular dg algebra $\mathcal{C}(\Lambda_K)$ can still be defined and computed with only slight modifications, see Section B.1. However, we will not need to deal with cone singularities in the main body of this thesis, since they have already been cancelled in the Legendrian front of $\Lambda_{p,q,r}$ using higher dimensional Reidemeister moves in the proof of Proposition 4.3.1.

Analogous to the surface case, we can consider the $K$-compatible polygonal decomposition of the real line $\mathbb{R}$. When $K$ has an arc component, this decomposition will include the origin of $\mathbb{R}$ as a 0-cell. For each cell $e$ (0-cell or 1-cell) in the cellular decomposition of $p_{x_1}(K)$, we can associate to it two cells $e$ and $\check{e}$ in the $\Lambda_K$-compatible polygonal decomposition of $\mathbb{R}^2$. The first one $e$ can be identified with the original cell through the embedding of the $x_1$-axis into $\mathbb{R}^2_{x_1,x_2}$. The second one $\check{e}$ is the spinning of $e$ around the $z$-axis. Note that $e$ and $\check{e}$ are the same if $e$ is the origin $\{x_1 = 0\}$, and we denote by $o$ the unique 0-cell it induces in the cellular decomposition of $p_x(\Lambda_K)$.

From the above we get two cellular dg algebras: one is associated to the cellular decomposition of $p_{x_1}(K)$, and the other one is associated to the $\Lambda_K$-compatible polygonal decomposition of $\mathbb{R}^2$ induced by the cellular decomposition of $p_{x_1}(K)$. We denote these dg algebras by $(\mathcal{C}(K), d_{\mathcal{C}})$ and $(\mathcal{C}(\Lambda_K), d^s_{\mathcal{C}})$ respectively. We remark that explicitly the differential $d_{\mathcal{C}}$ of the former cellular dg algebra $\mathcal{C}(K)$ is defined using (5.10) and (5.12). In fact, $\mathcal{C}(K)$ is quasi-isomorphic to the bordered Chekanov-Eliashberg algebra introduced by Sivek [104]. The way that we used to obtain the cellular decomposition of $p_x(\Lambda_K)$ suggests that algebraically these two dg algebras are related to each other through the suspension construction of Section 5.3. In fact, we have the following result.

**Proposition 5.4.1** ([85], Proposition 5.2). *Suppose $K \subset J^1(\mathbb{R})$ has arc components whose front projections have distinct endpoints, then the cellular dg algebra $(\mathcal{C}(\Lambda_K), d^s_{\mathcal{C}})$ is the suspension of $(\mathcal{C}(K), d_{\mathcal{C}})$ relative to the dg subalgebra $\mathcal{O}(K)$ associated to the 0-cell $e^0_o$ in the cellular decomposition of $p_x(\Lambda_K)$. If $K$ has two arc components whose front projections have the same end point, so that its front spin $\Lambda_K$ contains a unique cone singularity above $e^0_o$, $\mathcal{C}(\Lambda_K)$ is the suspension of $\mathcal{C}(K)$ relative to $\mathcal{O}(K)$ with the modification that in this case*

$$D(A_o) = \sum_{m<k} a_o^{m,k+1}\Delta_{m,k} + \sum_{k+1<n} a_o^{k,n}\Delta_{k+1,n} \tag{5.22}$$

*instead of $D(A_o) = 0$, where $A_o$ is the matrix of generators associated to the 0-cell $e^0_o$, and the cone point over $e^0_o$ connects the sheets $S_k$ and $S_{k+1}$.*

**Remark 5.4.1.** *When doing computations of* $\mathcal{C}(\Lambda_K)$ *using Proposition 5.4.1, one is supposed to start with a Maslov potential* $\mu_K : K \to \mathbb{Z}$ *on the Legendrian arc* $K$*, and then appeal to the grading convention of Definition 5.3.1 to determine the gradings on* $\mathcal{C}(\Lambda_K)$*. However, care must be taken as a disconnected Legendrian arc* $K$ *may become connected after spinning it around. As an example, we have the Legendrian surface* $\Lambda_{1,1,0}$ *obtained by spinning a two-component Legendrian arc* $K_{1,1,0}$*, see Figure 6.7. In this situation, we need to make sure that the Maslov potential* $\mu_K$ *that we use in the computation of the dg algebra* $\mathcal{C}(K)$ *is induced from some well-defined potential* $\mu_\Lambda : \Lambda_K \to \mathbb{Z}$*.*

# Chapter 6

# Computations of Chekanov-Eliashberg dg algebras

The main purpose of this chapter is to compute the Chekanov-Eliashberg dg algebras for the 2-dimensional Legendrian links $\Lambda_{p,q,r} \subset J^1(\mathbb{R}^2)$. For the computations in this chapter, we will work over $\mathbb{K} = \mathbb{Z}/2$, which enables us to adopt the cellular model introduced in the last chapter. As an illustration to more complicated computations, we will first compute in Section 6.1 the cellular dg algebra of the $A_r$ type Legendrian attaching link $\Lambda_r$ of standard unknots, which also enables us to reduce the computation for a general link $\Lambda_{p,q,r}$ to the special case when $p = q = r = 2$. The cellular dg algebra $\mathcal{C}(\Lambda_{2,2,2})$ will be computed in Section 6.2. For the links of Legendrian surfaces $\Lambda_r \subset \mathbb{R}^5$, their Legendrian fronts can be obtained by spinning a Legendrian arc in $\mathbb{R}^3$, so the algebraic construction in Section 5.4 will be used in order to simplify the computations.

## 6.1 The $A_r$ link

We compute the cellular dg algebra $\mathcal{C}(\Lambda_r)$ of the Legendrian link $\Lambda_r \subset (S^5, \xi_{std})$, which is the Legendrian front describing the 3-dimensional $A_r$ Milnor fiber

$$\{x^2 + y^2 + z^2 + w^{r+1} = 1\} \subset \mathbb{C}^4. \tag{6.1}$$

This is aimed at helping the readers understand our computation in the more complicated case in Section 6.2. Meanwhile, the computation here also provides an alternative proof of the Koszul duality result due to Ekholm-Lekili [36] for tree plumbings of $T^*S^3$'s in the special case when the plumbing tree $T = A_r$. The same method can be easily generalized to verify Koszul duality between the Fukaya $A_\infty$-algebras $\mathcal{F}_M$ and $\mathcal{W}_M$ of plumbings of $T^*S^3$'s according to any tree $T$.

Our proof uses the description of the Legendrian front of $\Lambda_r$ as the front spin of a Legendrian arc $K_r \subset J^1(\mathbb{R})$, see Figure 6.1. The computation of the cellular dg algebra $\mathcal{C}(\Lambda_r)$ can then be reduced to the computation of $\mathcal{C}(K_r)$ by Proposition 5.4.1.

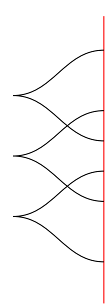

Figure 6.1: The Legendrian arc $K_3$, together with the axis $x_1 = 0$, coloured in red

Consider the Legendrian arc $K_r$, which consists of $r$ connected components, and crossings happen only between the nearby strands labelled by $i$ and $i + 1$. Without changing the Legendrian isotopy class, we can arrange that all the crossing points in the front of $K_r$ have the same base projection in $\mathbb{R}_{x_1}$. This violation of the genericity of the front projection is justified by the discussions in Section 5.2. A $K_r$-compatible polygonal decomposition associated to such a non-generic front projection of $K_r$ is depicted in Figure 6.2.

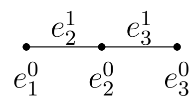


Figure 6.2: Cellular decomposition associated to $K_r$

Denote by $A_1, A_2, A_3, B_2, B_3$ the matrices of generators in $\mathcal{C}(K_r)$ associated to the cells $e_1^0, e_2^0, e_3^0, e_2^1, e_3^1$ in the above cellular decomposition. To start, we want to apply the formula (5.12) to the generators in the matrix $B_2$ associated to the 1-cell $e_2^1$. In this case, the role of matrix $A_{\alpha,-}$ in (5.12) is played by $A_1$, and the role of the matrix $A_{\alpha,+}$ is played by $A_2$. Since the cusps appear above the 0-cell $e_1^0$ are formed between the strands of $K_r$ labelled by

$$(1,2),(3,4),\dots,(2r-1,2r) \tag{6.2}$$

above the 1-cell $e_2^1$, by (5.16) we have

$$A_1 = \sum_{i=1}^{r} \Delta_{2i-1,2i}. \tag{6.3}$$

Since the strands of $K_r$ labelled by

$$(2,3),(4,5),\dots,(2r-2,2r-1) \tag{6.4}$$

above the 0-cell $e_2^0$ cross with each other, it follows by definition of the matrix $A_2$ recalled in Section 5.1 that

$$a_2^{2,3} = a_2^{4,5} = \cdots = a_2^{2r-2,2r-1} = 0. \tag{6.5}$$

For the other non-zero entries $a_2^{m,n}$ in the strictly upper-triangular matrix $A_2$, we can cancel them using the formula of $d_{\mathcal{C}}B_2$. More precisely, we have by (5.12) that

$$d_{\mathcal{C}}b_2^{m,n} = a_2^{m,n} + a_1^{m,n} + \sum_{m<k<n} a_2^{m,k}b_2^{k,n} + \sum_{m<k<n} b_2^{m,k}a_1^{k,n}. \tag{6.6}$$

By Lemma 5.3.1, we can define a filtration $F^\bullet$ on $\mathcal{C}(K_r)$ which respects the increasing order of $n-m$, and cancel the generators $b_2^{m,n}$ with $a_2^{m,n}$ according to the filtration $F^\bullet$ when $a_2^{m,n} \neq 0$. After the cancellation process, we get a quotient dg algebra of $\mathcal{C}(K_r)$ which is quasi-isomorphic to $\mathcal{C}(K_r)$. In particular, in the quotient dg algebra $\mathcal{C}'(K_r)$, the non-zero generators in $B_2$ are

$$b_2^{2,3}, b_2^{4,5}, \ldots, b_2^{2r-2,2r-1}, \tag{6.7}$$

and the identities

$$a_2^{m,n} = a_1^{m,n} + \sum_{m<k<n} a_2^{m,k}b_2^{k,n} + \sum_{m<k<n} b_2^{m,k}a_1^{k,n} \tag{6.8}$$

hold. In fact, for generators $b_2^{m,n}$ which has been cancelled with $a_2^{m,n}$, since they vanish in the quotient dg algebra $\mathcal{C}'(K_r)$, it is clear that we have (6.8) in $\mathcal{C}'(K_r)$ for any $(m,n) \neq (k,k+1)$, with $2 \leqslant k \leqslant 2r-2$. When $(m,n) = (k,k+1)$ for some $k$, it follows from (6.3) and (6.5) that $a_1^{k,k+1} = a_2^{k,k+1} = 0$, therefore by (6.6) we have $d_{\mathcal{C}}b_2^{k,k+1} = 0$, which shows that (6.8) is still true. Using (6.3) and (6.7), we can then compute $A_2$ inside the quotient dg algebra $\mathcal{C}'(K_r)$, and deduce that

$$A_2 = \begin{bmatrix} 0 & 1 & b_2^{2,3} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2^{2,3} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b_2^{4,5} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2^{4,5} & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & b_2^{2r-2,2r-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & b_2^{2r-2,2r-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \tag{6.9}$$

in $\mathcal{C}'(K_r)$.

Since there is no crossing above the 1-cell $e_3^1$, none of the entries above the diagonal of $A_3$ is zero in $\mathcal{C}(K_r)$. Arguing similarly as above, we see that all the generators in the matrix $B_3$ can be cancelled with the generators in $A_3$. In $\mathcal{C}'(K_r)$, we have the following identity:

$$a_3^{m,n} = a_2^{\sigma^{-1}(m),\sigma^{-1}(n)} + \sum_{m<k<n} a_3^{m,k} b_3^{k,n} + \sum_{m<k<n} b_3^{m,k} a_2^{\sigma^{-1}(k),\sigma^{-1}(n)}, \tag{6.10}$$

where it follows from our discussions in Section 5.2 that the permutation $\sigma$ is given by

$$\sigma = (2,3)(4,5)\dots(2r-2,2r-1). \tag{6.11}$$

It follows that in $\mathcal{C}'(K_r)$,

$$A_3 = \begin{bmatrix} 0 & b_2^{2,3} & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2^{4,5} & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{2,3} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2^{6,7} & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_2^{4,5} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_2^{2r-2,2r-1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & b_2^{2r-4,2r-3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & b_2^{2r-2,2r-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \tag{6.12}$$

By Proposition 5.4.1, the cellular dg algebra $\mathcal{C}(\Lambda_r)$ is the suspension of $\mathcal{C}(K_r)$ relative to the dg subalgebra $\mathcal{O}(K_r)$ generated by generators in $A_3$. Denote by $D_r : \mathcal{C}(K_r) \to \mathcal{C}(\Lambda_r)$ the derivation appearing in the definition of the suspension of $\mathcal{C}(K_r)$, which acts on the generators by $D_r(a_i^{m,n}) = \check{a}_i^{m,n}$ and $D_r(b_j^{m,n}) = \check{b}_j^{m,n}$. Further cancellations among the generators in $\mathcal{C}(\Lambda_r)$ imply

**Proposition 6.1.1.** $\mathcal{C}(\Lambda_r)$ *is quasi-isomorphic to a dg algebra generated by*

$$\begin{aligned} &b_2^{2,3}, b_2^{4,5}, \dots, b_2^{2r-2,2r-1}, \\ &\check{b}_3^{1,3}, \check{b}_3^{2,5}, \check{b}_3^{4,7}, \dots, \check{b}_3^{2r-4,2r-1}, \check{b}_3^{2r-2,2r}, \\ &\check{b}_3^{2,3}, \check{b}_3^{4,5}, \dots, \check{b}_3^{2r-2,2r-1}. \end{aligned} \tag{6.13}$$

*Proof.* We have already seen how the generators in $\mathcal{C}(K_r)$ are cancelled, so it suffices to consider which one of the generators $\check{b}_2^{m,n}$ and $\check{b}_3^{m,n}$ remains in the quotient dg algebra $\mathcal{C}(\Lambda_r)$ after cancellation. Applying the derivation $D_r$ to (6.6) we see that all the generators $\check{b}_2^{m,n}$ can be cancelled with $\check{a}_2^{m,n}$ except for

$$\check{b}_2^{2,3}, \check{b}_2^{4,5}, \ldots, \check{b}_2^{2r-2,2r-1}. \tag{6.14}$$

Using the fact that $\check{a}_3^{m,n} = 0$ and $b_3^{m,n} = 0$, $D_r$ applied to (6.10) implies that

$$d_{\mathcal{C}}\check{b}_3^{m,n} = \check{a}_2^{\sigma^{-1}(m),\sigma^{-1}(n)} + \sum_{m<k<n} a_3^{m,k}\check{b}_3^{k,n} + \sum_{m<k<n} \check{b}_3^{m,k}a_2^{\sigma^{-1}(k),\sigma^{-1}(n)}. \tag{6.15}$$

In the quotient dg algebra $\mathcal{C}'(\Lambda_r)$, the values of $a_3^{m,k}$ and $a_2^{\sigma^{-1}(k),\sigma^{-1}(n)}$ in the above formula have been determined in (6.9) and (6.12), from which we see that the generators in (6.14) can be cancelled with

$$\check{b}_3^{1,2}, \check{b}_3^{2,4}, \ldots, \check{b}_3^{2r-4,2r-2}. \tag{6.16}$$

On the other hand, using the entries of $A_3$ which are equal to 1 in $\mathcal{C}'(\Lambda_r)$ as specified by (6.12), we can cancel most of the generators of the form $\check{b}_3^{m,n}$ with each other as follows.

Since $a_3^{1,3} = 1$, by (6.15) we can cancel the following generators with each other:

$$(\check{b}_3^{1,4}, \check{b}_3^{3,4}), (\check{b}_3^{1,5}, \check{b}_3^{3,5}), \ldots, (\check{b}_3^{1,2r}, \check{b}_3^{3,2r}). \tag{6.17}$$

It follows immediately that the only remaining generator in the first row of $\check{B}_3$ in the quotient dg algebra $\mathcal{C}'(\Lambda_r)$ is $\check{b}_3^{1,3}$.

Similarly, since $a_3^{2,5} = a_3^{4,7} = \cdots = a_3^{2r-4,2r-1} = 1$, we get the following cancelling pairs of generators in $\mathcal{C}(\Lambda_r)$:

$$\begin{aligned}&(\check{b}_3^{1,2j+3}, \check{b}_3^{1,2j}), (\check{b}_3^{2,2j+3}, \check{b}_3^{2,2j}), \cdots, (\check{b}_3^{2j-1,2j+3}, \check{b}_3^{2j-1,2j}),\\ &(\check{b}_3^{2j,2j+4}, \check{b}_3^{2j+3,2j+4}), (\check{b}_3^{2j,2j+5}, \check{b}_3^{2j+3,2j+5}), \cdots, (\check{b}_3^{2j,2r}, \check{b}_3^{2j+3,2r}),\end{aligned} \tag{6.18}$$

where $j \leqslant r-2$. For any integer $k$ with $1 \leqslant k \leqslant r-1$, by considering the cancellations between $\check{b}_3^{2k,2j+3}$ and $\check{b}_3^{2k,2j}$ for $k+1 \leqslant j \leqslant r-2$, we see that in the $2k$-th row of $\check{B}_3$, only

$$\check{b}_3^{2k,2k+1}, \check{b}_3^{2k,2k+3}, \check{b}_3^{2k,2r-2}, \check{b}_3^{2k,2r} \tag{6.19}$$

remain in $\mathcal{C}'(\Lambda_r)$. On the other hand, the cancellation pairs $(\check{b}_3^{2k-1,2j+3}, \check{b}_3^{2k-1,2j})$ in the above list shows that the only remaining generators in the $(2k-1)$-th row of $\check{B}_3$ are given by

$$\check{b}_3^{2k-1,2k+1}, \check{b}_3^{2k-1,2r}. \tag{6.20}$$

Finally, using the fact that $a_3^{2r-2,2r} = 1$, it is not hard to see that we can further cancel the generators

$$(\check{b}_3^{2k,2r-2}, \check{b}_3^{2k,2r}), (\check{b}_3^{2k-1,2k+1}, \check{b}_3^{2k-1,2r}), \tag{6.21}$$

which completes the proof. □

To compute the gradings of the generators listed in (6.13), we need to specify a Maslov potential on $\Lambda_r$. In our case, it suffices to define a Maslov potential $\mu_r : K_r \to \mathbb{Z}$ on the Legendrian arc $K_r$, and then apply the grading formula in Definition 5.3.1. Denote by $S_1, \ldots, S_{2r}$ the strands of $K_r$ above the 1-cell $e_3^1$, we can define $\mu_r$ by setting

$$\mu_n(S_{2r}) = 0, \mu_n(S_{2r-1}) = \mu_n(S_{2r-2}) = 1, \ldots, \mu_n(S_3) = \mu_n(S_2) = n-1, \mu(S_1) = n. \tag{6.22}$$

This implies that

$$|b_2^{2,3}| = |b_2^{4,5}| = \cdots = |b_2^{2r-2,2r-1}| = 0, \tag{6.23}$$

$$|\check{b}_3^{2,3}| = |\check{b}_3^{4,5}| = \cdots = |\check{b}_3^{2r-2,2r-1}| = -1, \tag{6.24}$$

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \quad \cdots \quad \longrightarrow \bullet$$

Figure 6.3: $A_r$ quiver

and

$$|\check{b}_3^{2,3}| = |\check{b}_3^{4,5}| = \cdots = |\check{b}_3^{2r-2,2r-1}| = -2. \tag{6.25}$$

It remains to compute the differentials $d_{\mathcal{C}}$ on the generators of $\mathcal{C}'(\Lambda_r)$. From the cancellation procedure of the generators in $\mathcal{C}(K_r)$ we can extract the following:

$$d_{\mathcal{C}} b_2^{2,3} = d_{\mathcal{C}} b_2^{4,5} = \cdots = d_{\mathcal{C}} b_2^{2r-2,2r-1} = 0. \tag{6.26}$$

Using (6.15) it is easy to deduce that

$$d_{\mathcal{C}} \check{b}_3^{2,3} = d_{\mathcal{C}} \check{b}_3^{4,5} = \cdots = d_{\mathcal{C}} \check{b}_3^{2r-2,2r-1} = 0, \tag{6.27}$$

and

$$d_{\mathcal{C}} \check{b}_3^{1,3} = b_2^{2,3} \check{b}_3^{2,3}, d_{\mathcal{C}} \check{b}_3^{2r-2,2r} = \check{b}_3^{2r-2,2r-1} b_2^{2r-2,2r-1}, \tag{6.28}$$

$$d_{\mathcal{C}} \check{b}_3^{2j,2j+3} = \check{b}_3^{2j,2j+1} b_2^{2j,2j+1} + b_2^{2j+2,2j+3} \check{b}_3^{2j+2,2j+3}, \tag{6.29}$$

for any $j$ with $1 \leqslant j \leqslant r-2$.

The computations above can be summarized as follows:

**Proposition 6.1.2.** *The cellular dg algebra $\mathcal{C}(\Lambda_r)$ is quasi-isomorphic to the Ginzburg dg algebra associated to the $A_r$ quiver depicted in Figure 6.3.*

**Remark 6.1.1.** *The orientation data of the holomorphic polygons involved in the definition of the Chekanov-Eliashberg algebra $CE^*(\Lambda_r)$ can be recovered by applying directly the Koszul duality result of Ekholm-Lekili [36], or by identifying the differentials in $\mathcal{C}'(\Lambda_r)$ with the enumerations of Morse flow trees, and then applying the machinery recalled in Appendix A.*

## 6.2 The link $\Lambda_{2,2,2}$

This section is devoted to the computation of the cellular dg algebra of the link of Legendrian surfaces $\Lambda_{2,2,2} \subset J^1(\mathbb{R}^2)$, whose front projection $p_{x,z}(\Lambda_{2,2,2})$ is described by Figure 4.5. Note that it is not hard to get from Figure 4.5 the Legendrian front of the general link $\Lambda_{p,q,r}$, by replacing the standard unknot $\Lambda_P$ with an $A_{p-1}$-chain of parallel unknots, $\Lambda_Q$ with an $A_{q-1}$-chain of unknots, and $\Lambda_R$ with an $A_{r-1}$-chain of unknots. As we have remarked at the beginning of this chapter, in order to compute $\mathcal{C}(\Lambda_{p,q,r})$, it suffices to compute $\mathcal{C}(\Lambda_{2,2,2})$, and then combine the computations here with that in Section 6.1.

A $\Lambda_{2,2,2}$-compatible polygonal decomposition of $p_x(\Lambda_{2,2,2})$ associated to the Legendrian front depicted in Figure 4.5 is given in Figure 6.4. In that figure, the 1-cells associated to cusp edges are represented by solid curves, and the 1-cells associated to crossing arcs are represented by dashed arcs. As a convention, all the cells $e_i^0, e_j^1$ and $e_k^2$ in the polygonal decomposition are labelled by their associated matrices of generators $A_i, B_j$ and $C_k$, respectively. Note that above the largest solid circle in Figure 6.4, one can find only the cusp edges of the components $\Lambda_A, \Lambda_B$ and $\Lambda_R$, so there is no generator in $\mathcal{C}(\Lambda_{2,2,2})$ associated to it. In particular, the Legendrian front we are using for $\Lambda_{2,2,2}$ is non-generic, and such a choice is justified by our discussions in Section 5.2. Besides the projection $\Sigma_{2,2,2} \subset \mathbb{R}^2$ of the set of singularities of $p_{x,z}(\Lambda_{2,2,2})$, we have added in the cellular decomposition the 0-cells labelled by the matrices

$$A_1, \ldots, A_{12}; \tag{6.30}$$

and the 1-cells labelled by the matrices

$$B_2, \ldots, B_7, B_9, \ldots, B_{12}, \tag{6.31}$$

so that the non-simply-connected regions in $p_x(\Lambda_{2,2,2})$ are divided into polygons after these cells are added. All the other 1-cells correspond to singular loci in the Legendrian front of $\Lambda_{2,2,2}$, and their geometric meanings are recorded in the following table.

| Labelling | Front above the 1-cell |
|---|---|
| $B_1, B_8$ | $p_{x,z}(\Lambda_A) \cap p_{x,z}(\Lambda_B)$ |
| $B_{13}$ | $p_{x,z}(\Lambda_B) \cap p_{x,z}(\Lambda_P)$ |
| $B_{14}$ | $p_{x,z}(\Lambda_A) \cap p_{x,z}(\Lambda_P)$ |
| $B_{15}$ | cusp edge of $\Lambda_P$ |
| $B_{16}$ | $p_{x,z}(\Lambda_B) \cap p_{x,z}(\Lambda_Q)$ |
| $B_{17}$ | $p_{x,z}(\Lambda_A) \cap p_{x,z}(\Lambda_Q)$ |
| $B_{18}$ | cusp edge of $\Lambda_Q$ |
| $B_{19}, B_{20}$ | $p_{x,z}(\Lambda_B) \cap p_{x,z}(\Lambda_R)$ |
| $B_{21}$ | $p_{x,z}(\Lambda_A) \cap p_{x,z}(\Lambda_R)$ |

Denote by $J = \{1, \ldots, 21\}$ the set of subscripts of the matrices $B_j$ associated to the 1-cells in the cellular decomposition of $p_x(\Lambda_{2,2,2})$ specified above. Let

$$J_1 = \{2, \ldots, 7, 9, \ldots, 12, 19, 20\} \subset J. \tag{6.32}$$

For any 1-cell $e^1_j$ with $j \in J_1$, it has two distinct endpoints, which are labelled as the 0-cells $e^0_+$ and $e^0_-$, by (5.12) we have

$$d_{\mathcal{C}} b_j^{m,n} = a_+^{m,n} + a_-^{\sigma_j(m),\sigma_j(n)} + \sum_{m<k<n} a_+^{m,k} b_j^{k,n} + \sum_{m<k<n} b_j^{m,k} a_-^{\sigma_j(k),\sigma_j(n)}, \tag{6.33}$$

where the permutation $\sigma_j$ is a transposition determined by the crossing arc of $p_{x,z}(\Lambda_{2,2,2})$ above the 0-cell $e^0_-$.

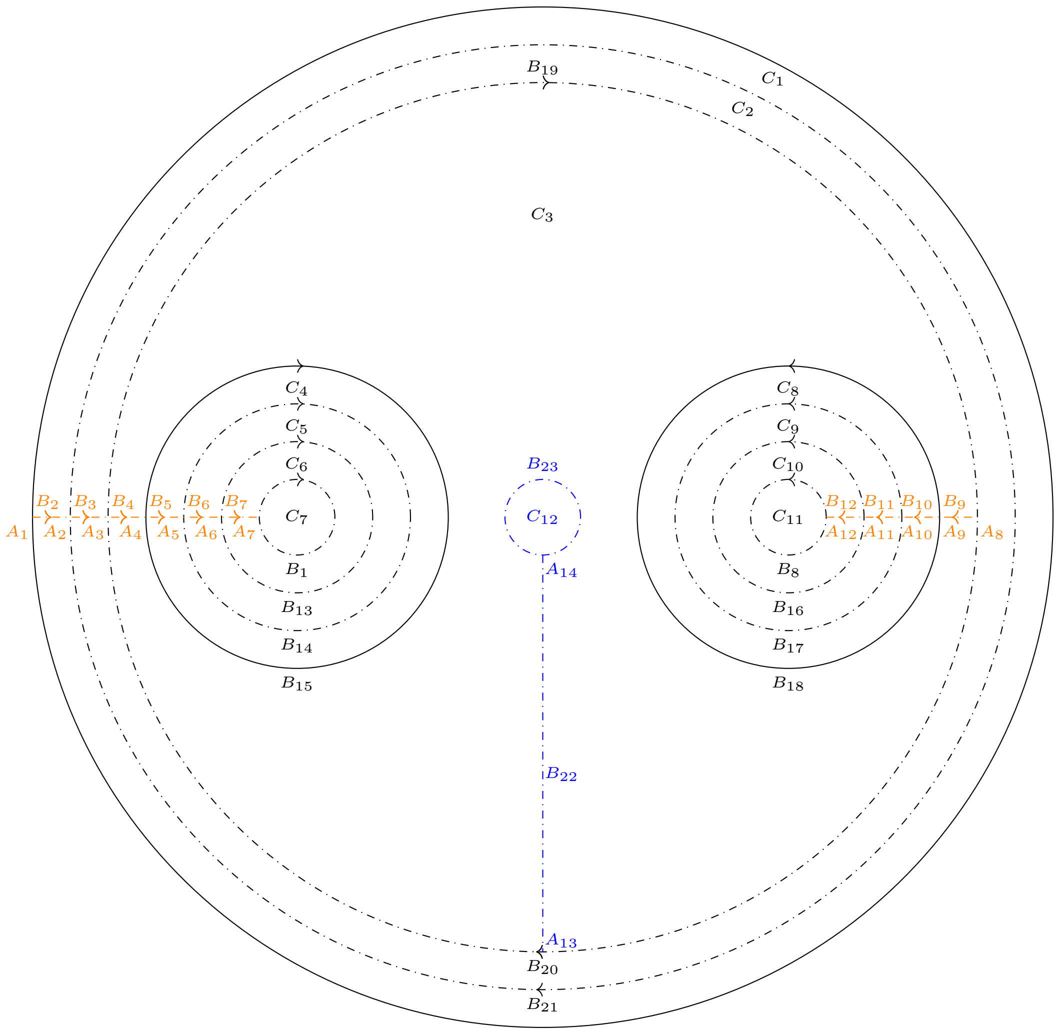


Figure 6.4: Cellular decomposition associated to $\Lambda_{2,2,2}$ (coloured in black and orange) and $\Lambda_{p,q,r}$ (with additional cells coloured in blue)

As we have already seen in Section 6.1, one can cancel the generators $b_j^{m,n}$ with $a_j^{m,n}$ using the formulas in (6.33). In particular, we see that the remaining generators in $B_j$ with $j \in J_1$ in the quotient dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$ are

$$b_2^{4,5}, b_3^{2,3}, b_5^{5,6}, b_6^{3,4}, b_7^{4,5}, b_{10}^{5,6}, b_{11}^{3,4}, b_{12}^{4,5}, \tag{6.34}$$

together with all the generators in the strictly upper triangular matrix $B_{20}$ except for $b_{20}^{2,3}$, which is equal to 0 by definition.

Let $J_2 = J \backslash J_1$. The 1-cells $e_j^1$ with $j \in J_2$ have the same 0-cell as their initial and terminal point, whose associated matrix of generators will be denoted by $A_{t_j}$. By (5.12), the differentials of generators in $B_j$ are given by

$$d_{\mathfrak{C}} b_j^{m,n} = \sum_{m<k<n} a_{t_j}^{m,k} b_j^{k,n} + \sum_{m<k<n} b_j^{m,k} a_{t_j}^{k,n}. \tag{6.35}$$

In particular, these generators are not cancelled with any generators associated to 0-cells.

For the index set $I$ of the 2-cells in the cellular decomposition of $\Lambda_{2,2,2}$, denote by $I_1 \subset I$ the subset

$$I_1 = \{1, 4, 5, 6, 8, 9, 10\}. \tag{6.36}$$

For any $i \in I_1$, the corresponding 2-cell $e_i^2$ is an annulus bounded by two circles, whose inner circle has generators assembled in the matrix $B_{s_i}$ and whose outer circle is labelled by the matrix $B_{t_i}$, together with an additional cutting edge $e_{r_i}^1$, whose endpoints have associated matrices of generators $A_{i,-}$ and $A_{i,+}$, see Figure 6.5. By (5.15) we can write down the formulas of their differentials:

$$d_{\mathfrak{C}} C_i = A_{i,+} C_i + C_i Q_{\sigma_{r_i}} A_{i,-} Q_{\sigma_{r_i}} + B_{s_i}(E + B_{r_i}) + (E + B_{r_i}) Q_{\sigma_{r_i}} B_{t_i} Q_{\sigma_{r_i}}, \tag{6.37}$$

where

$$Q_{\sigma_{r_i}} = \sum_m \Delta_{\sigma_{r_i}(m),m} \tag{6.38}$$

is the permutation matrix. Note that when $i = 1$, $B_{t_1} = 0$ in the above formula. In particular, (6.37) shows that all the generators in $C_i$ with $i \in I_1$ can be cancelled with that in $B_{s_i}$, except for

$$c_1^{4,5}, c_4^{5,6}, c_5^{3,4}, c_6^{4,5}, c_8^{5,6}, c_9^{3,4}, c_{10}^{4,5}. \tag{6.39}$$

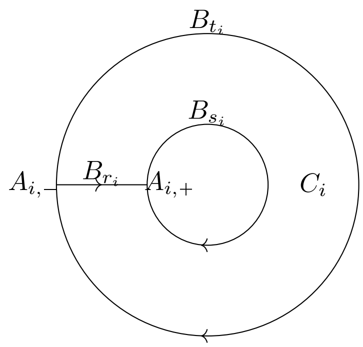


Figure 6.5: A type $I_1$ 2-cell labelled by $C_i$

Similarly, the 2-cell $e_2^2$ is also an annulus, with the cutting edge $e_3^1$. The only difference is that the small circle which bounds the annulus $e_2^2$ now consists of two 1-cells, namely $e_{19}^1$ and $e_{20}^1$, and the large circle has the associated 1-cell $e_{21}^1$. As a consequence,

$$d_{\mathcal{C}}C_2 = A_3C_2 + C_2Q_{\sigma_3}A_2Q_{\sigma_3} + (B_{19} + B_{20} + B_{20}B_{19})(E + B_3) + (E + B_3)Q_{\sigma_3}B_{21}Q_{\sigma_3}. \tag{6.40}$$

Since $B_{19} = 0$ after cancelling it with $A_8$, the above formula simplifies to

$$d_{\mathcal{C}}C_2 = A_3C_2 + C_2Q_{\sigma_3}A_2Q_{\sigma_3} + B_{20}(E + B_3) + (E + B_3)Q_{\sigma_3}B_{21}Q_{\sigma_3} \tag{6.41}$$

in $\mathcal{C}'(\Lambda_{2,2,2})$. Since $b_{20}^{2,3} = 0$, we see that the generator $c_2^{2,3}$ is the only generator in the matrix $C_2$ that cannot be cancelled via the above formula.

On the generators in $C_7$ and $C_{11}$, differential $d_{\mathcal{C}}$ takes the form

$$d_{\mathcal{C}}C_7 = Q_0A_7Q_0C_7 + C_7Q_0A_7Q_0 + Q_0B_1Q_0 \tag{6.42}$$

and

$$d_{\mathcal{C}}C_{11} = Q_0A_{12}Q_0C_{11} + C_{11}Q_0A_{12}Q_0 + Q_0B_8Q_0, \tag{6.43}$$

where $Q_0$ is the permutation matrix associated to the transposition $\sigma_0 = (4, 5)$.

For the remaining 2-cell labelled by $e^2_3$, its boundary consists of the 1-cells

$$e^1_4, e^1_9, e^1_{15}, e^1_{18}, e^1_{19}, e^1_{20}. \tag{6.44}$$

Choosing $e^0_4$ as the initial vertex, and $e^0_9$ as the terminal vertex, we have

$$\begin{aligned} d_{\mathcal{C}} C_3 = {} & A_9 C_3 + C_3 A_4 + (E + B_9)(E + Q_{\sigma_4} B_{19} Q_{\sigma_4})(E + B_4)^{-1}(E + B_{15}) \\ & + (E + B_{18})(E + B_9)(E + Q_{\sigma_4} B_{20} Q_{\sigma_4})^{-1}(E + B_4)^{-1}. \end{aligned} \tag{6.45}$$

Since $B_4 = B_9 = B_{19} = 0$ after cancelling their generators with $A_4, A_9$ and $A_8$, the above formula simplifies to

$$d_{\mathcal{C}} C_3 = A_9 C_3 + C_3 A_4 + E + B_{15} + (E + B_{18})(E + Q_{\sigma_4} B_{20} Q_{\sigma_4})^{-1}. \tag{6.46}$$

In particular, all the generators in $C_3$ are cancelled with that of $B_{15}$.

**Proposition 6.2.1.** *The cellular dg algebra $\mathcal{C}(\Lambda_{2,2,2})$ is quasi-isomorphic to a dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$ generated by*

$$\begin{gathered} b_2^{4,5}, b_3^{2,3}, b_5^{5,6}, b_6^{3,4}, b_7^{4,5}, b_{10}^{5,6}, b_{11}^{3,4}, b_{12}^{4,5}; \\ c_7^{4,5}, c_{11}^{4,5}, c_7^{3,5}, c_{11}^{3,5}, c_7^{2,5} + c_{11}^{2,5}, c_7^{4,6}, c_{11}^{4,6}, c_7^{4,7} + c_{11}^{4,7}; \\ c_7^{4,8} + c_{11}^{4,8}, c_7^{1,5} + c_{11}^{1,5}, c_7^{3,6}, c_{11}^{3,6}, c_7^{2,7} + c_{11}^{2,7}, \end{gathered} \tag{6.47}$$

*with gradings*

$$\begin{aligned} & |b_2^{4,5}| = |b_3^{2,3}| = |b_5^{5,6}| = |b_6^{3,4}| = |c_7^{4,5}| = |b_{10}^{5,6}| = |b_{11}^{3,4}| = |c_{11}^{4,5}| = 0, \\ & |b_7^{4,5}| = |b_{12}^{4,5}| = |c_7^{3,5}| = |c_{11}^{3,5}| = |c_7^{2,5} + c_{11}^{2,5}| = |c_7^{4,6}| = |c_{11}^{4,6}| = |c_7^{4,7} + c_{11}^{4,7}| = -1, \\ & |c_7^{4,8} + c_{11}^{4,8}| = |c_7^{1,5} + c_{11}^{1,5}| = |c_7^{3,6}| = |c_{11}^{3,6}| = |c_7^{2,7} + c_{11}^{2,7}| = -2, \end{aligned} \tag{6.48}$$

*and differentials*

$$d_{\mathcal{C}} b_2^{4,5} = d_{\mathcal{C}} b_3^{2,3} = d_{\mathcal{C}} b_5^{5,6} = d_{\mathcal{C}} b_6^{3,4} = d_{\mathcal{C}} c_7^{4,5} = d_{\mathcal{C}} b_{10}^{5,6} = d_{\mathcal{C}} b_{11}^{3,4} = d_{\mathcal{C}} c_{11}^{4,5} = 0, \tag{6.49}$$

$$d_{\mathcal{C}} b_7^{4,5} = b_6^{3,4} b_5^{5,6} + b_3^{2,3} b_2^{4,5}, \tag{6.50}$$

$$d_{\mathcal{C}} b_{12}^{4,5} = b_{11}^{3,4} b_{10}^{5,6} + b_3^{2,3} b_2^{4,5}, \tag{6.51}$$

$$d_{\mathcal{C}} c_7^{3,5} = b_5^{5,6} c_7^{4,5}, \tag{6.52}$$

$$d_{\mathcal{C}} c_{11}^{3,5} = b_{10}^{5,6} c_{11}^{4,5}, \tag{6.53}$$

$$d_{\mathcal{C}} (c_7^{2,5} + c_{11}^{2,5}) = b_2^{4,5} c_{11}^{4,5} + b_2^{4,5} c_7^{4,5}, \tag{6.54}$$

$$d_{\mathcal{C}} c_7^{4,6} = c_7^{4,5} b_6^{3,4}, \tag{6.55}$$

$$d_{\mathcal{C}} c_{11}^{4,6} = c_{11}^{4,5} b_{11}^{3,4}, \tag{6.56}$$

$$d_{\mathcal{C}} (c_7^{4,7} + c_{11}^{4,7}) = c_{11}^{4,5} b_3^{2,3} + c_7^{4,5} b_3^{2,3}, \tag{6.57}$$

$$d_{\mathcal{C}} (c_7^{4,8} + c_{11}^{4,8}) = c_{11}^{4,5} b_{12}^{4,5} + c_7^{4,5} b_7^{4,5} + c_7^{4,6} b_5^{5,6} + c_{11}^{4,6} b_{10}^{5,6} + (c_7^{4,7} + c_{11}^{4,7}) b_2^{4,5}, \tag{6.58}$$

$$d_{\mathcal{C}} (c_7^{1,5} + c_{11}^{1,5}) = b_6^{3,4} c_7^{3,5} + b_{11}^{3,4} c_{11}^{3,5} + b_3^{2,3} (c_7^{2,5} + c_{11}^{2,5}) + b_{12}^{4,5} c_{11}^{4,5} + b_7^{4,5} c_7^{4,5}, \tag{6.59}$$

$$d_{\mathcal{C}} c_7^{3,6} = b_5^{5,6} c_7^{4,6} + c_7^{3,5} b_6^{3,4}, \tag{6.60}$$

$$d_{\mathcal{C}} c_{11}^{3,6} = b_{10}^{5,6} c_{11}^{4,6} + c_{11}^{3,5} b_{11}^{3,4}, \tag{6.61}$$

$$d_{\mathcal{C}} (c_7^{2,7} + c_{11}^{2,7}) = b_2^{4,5} (c_7^{4,7} + c_{11}^{4,7}) + (c_7^{2,5} + c_{11}^{2,5}) b_3^{2,3}. \tag{6.62}$$

*Proof.* Turning back to the differentials $d_{\mathcal{C}} b_j^{m,n}$ for $j \in J_1$, we need to record the values of $A_i = (a_i^{m,n})$ in the quotient dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$ after the cancellation process between generators.

Since above the 0-cell $e_1^0$ are the cusp edges of the Legendrian fronts of the components $\Lambda_A$, $\Lambda_B$ and $\Lambda_R$ of the link $\Lambda_{2,2,2}$, by (5.16), $A_1$ is a block diagonal matrix with three $N$ blocks on its diagonal, i.e.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.63}$$

After cancellation, $b_2^{4,5}$ is the only remaining generator in $B_2$, from this fact and (6.33) we get

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & b_2^{4,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.64}$$

The only remaining generator in $B_3$ after cancelling it with $A_3$ is $b_3^{2,3}$, combining with $\sigma_3 = (4,5)$ we obtain

$$A_3 = \begin{bmatrix} 0 & 1 & b_3^{2,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3^{2,3} b_2^{4,5} & b_3^{2,3} & 0 \\ 0 & 0 & 0 & b_2^{4,5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & b_2^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.65}$$

$B_4 = 0$ after cancelling it with the generators in $A_4$. Since there is a cusp edge of $p_{x,z}(\Lambda_P)$ above the 0-cell $e_5^0$, in addition to applying the transposition $\sigma_4 = (2,3)$, we need to add an $N$ block in the fourth and fifth rows in order to obtain $A_4$ in $\mathcal{C}'(\Lambda_{2,2,2})$. This gives:

$$A_4 = \begin{bmatrix} 0 & b_3^{2,3} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2^{4,5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3^{2,3}b_2^{4,5} & b_3^{2,3} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.66}$$

To determine the value of $A_5$ in $\mathcal{C}'(\Lambda_{2,2,2})$, we notice that the $b_5^{5,6}$ remains after cancelling $B_5$ with $A_5$:

$$A_5 = \begin{bmatrix} 0 & b_3^{2,3} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_2^{4,5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_3^{2,3}b_2^{4,5} & b_3^{2,3} & 0 \\ 0 & 0 & 0 & 0 & 1 & b_5^{5,6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_5^{5,6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.67}$$

$b_6^{3,4}$ is the only generator in the matrix $B_6$ that is not cancelled with the generators in $A_6$. As $\sigma_6 = (5,6)$, it enables us to compute:

$$A_6 = \begin{bmatrix} 0 & b_3^{2,3} & 1 & b_6^{3,4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{4,5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_3^{2,3}b_2^{4,5} + b_6^{3,4}b_5^{5,6} & b_6^{3,4} & b_3^{2,3} & 0 \\ 0 & 0 & 0 & 0 & b_5^{5,6} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_5^{5,6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.68}$$

Finally, note that $b_7^{4,5}$ can not be cancelled with any of the generators in $A_7$. Since $\sigma_7 = (3,4)$, it follows that

$$A_7 = \begin{bmatrix} 0 & b_3^{2,3} & b_6^{3,4} & 1 & b_7^{4,5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{4,5} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & b_5^{5,6} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_6^{3,4} & b_3^{2,3} & b_7^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_5^{5,6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.69}$$

By (6.33) applied to the case when $j = 7$, we deduce from the computations for $A_6$ and $A_7$ above that

$$\boxed{d_{\mathcal{C}} b_7^{4,5} = b_3^{2,3}b_2^{4,5} + b_6^{3,4}b_5^{5,6}.} \tag{6.70}$$

**Remark 6.2.1.** *As a convention, we will box the formulas which contribute to the non-trivial differentials of the generators in the dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$.*

Applying (6.37) when $i = 1$ implies that

$$B_{21} = A_2C_1 + C_1A_1 + B_{21}B_2 \tag{6.71}$$

holds in $\mathcal{C}'(\Lambda_{2,2,2})$. From this one deduces that

$$B_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_2^{4,5}c_1^{4,5} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.72}$$

Since all the generators in $C_2$ except for $c_2^{2,3}$ have been cancelled in $\mathcal{C}'(\Lambda_{2,2,2})$, it follows from (6.41) that

$$B_{20} = Q_{\sigma_3}B_{21}Q_{\sigma_3} + A_3C_2 + C_2Q_{\sigma_3}A_2Q_{\sigma_3} + B_{20}B_3 + B_3Q_{\sigma_3}B_{21}Q_{\sigma_3}. \tag{6.73}$$

Combining with (6.72), (6.64) and (6.65), one can deduce that

$$B_{20} = \begin{bmatrix} 0 & 0 & c_2^{2,3} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2^{2,3}b_2^{4,5} + b_3^{2,3}b_2^{4,5}c_1^{4,5} & c_2^{2,3} & 0 \\ 0 & 0 & 0 & b_2^{4,5}c_1^{4,5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_1^{4,5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.74}$$

It is straightforward to verify that

$$(E + Q_{\sigma_4}B_{20}Q_{\sigma_4})^{-1} = E + Q_{\sigma_4}B_{20}Q_{\sigma_4}. \tag{6.75}$$

Since $C_3 = 0$ in $\mathcal{C}'(\Lambda_{2,2,2})$, we get from (6.46) that

$$B_{15} + B_{18} = Q_{\sigma_4} B_{20} Q_{\sigma_4} + B_{18} Q_{\sigma_4} B_{20} Q_{\sigma_4}. \tag{6.76}$$

When computing $d_{\mathcal{C}} C_4$, since the cusp edge of $\Lambda_P$ lies above the 1-cell with labelling $B_{15}$, by (5.14) we need to replace $B_{15}$ with an $8 \times 8$ matrix $\widetilde{B}_{15}$ by adding a $2 \times 2$ zero block in the fourth and fifth rows and columns of $B_{15}$. By (6.37) it follows that

$$B_{14} = \widetilde{B}_{15} + A_5 C_4 + C_4 A_4 + B_{14} B_5 + B_5 \widetilde{B}_{15}, \tag{6.77}$$

Applying (6.37) when $i = 5, 6$ implies that

$$B_{13} = Q_{\sigma_6} B_{14} Q_{\sigma_6} + A_6 C_5 + C_5 Q_{\sigma_6} A_5 Q_{\sigma_6} + B_{13} B_6 + B_6 Q_{\sigma_6} B_{14} Q_{\sigma_6} \tag{6.78}$$

and

$$B_1 = Q_{\sigma_7} B_{13} Q_{\sigma_7} + A_7 C_6 + C_6 Q_{\sigma_7} A_6 Q_{\sigma_7} + B_1 B_7 + B_7 Q_{\sigma_7} B_{13} Q_{\sigma_7} \tag{6.79}$$

hold in the quotient dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$.

From now on we compute the differentials of the generators in $C_7$ according to the filtration $F^\bullet$ on the cellular dg algebra $\mathcal{C}(\Lambda_{2,2,2})$ defined according to the increasing order of $n - m$.

For $n = m + 1$, by (6.42) and the cancellations made above we see that the differentials in $\mathcal{C}'(\Lambda_{2,2,2})$ are

$$d_{\mathcal{C}} c_7^{1,2} = b_{18}^{1,2}, \tag{6.80}$$

$$d_{\mathcal{C}} c_7^{2,3} = b_{18}^{2,3} b_6^{3,4}, \tag{6.81}$$

$$d_{\mathcal{C}} c_7^{3,4} = c_4^{5,6}, \tag{6.82}$$

$$\boxed{d_{\mathcal{C}} c_7^{4,5} = 0,} \tag{6.83}$$

$$d_{\mathcal{C}} c_7^{5,6} = c_5^{3,4}, \tag{6.84}$$

$$d_{\mathcal{C}} c_7^{6,7} = b_5^{5,6} b_{18}^{4,5}, \tag{6.85}$$

$$d_{\mathcal{C}} c_7^{7,8} = c_1^{4,5} + b_{18}^{5,6}. \tag{6.86}$$

In particular, by (6.82), (6.84) and (6.86) the generators

$$(c_7^{3,4}, c_4^{5,6}), (c_7^{5,6}, c_5^{3,4}), (c_7^{7,8}, c_1^{4,5}) \tag{6.87}$$

can be cancelled with each other, and $C_4 = C_5 = 0$ in $\mathcal{C}'(\Lambda_{2,2,2})$. Thus one can further simplify (6.77) and (6.78) respectively to

$$B_{14} = \widetilde{B}_{15} + B_{14} B_5 + B_5 \widetilde{B}_{15} \tag{6.88}$$

and

$$B_{13} = Q_{\sigma_6} B_{14} Q_{\sigma_6} + B_{13} B_6 + B_6 Q_{\sigma_6} B_{14} Q_{\sigma_6}. \tag{6.89}$$

For $n = m + 2$, taking into account of the above computations, one can deduce

$$d_{\mathcal{C}} c_7^{1,3} = b_3^{2,3} c_7^{2,3} + b_{18}^{1,3} b_6^{3,4}, \tag{6.90}$$

$$d_{\mathcal{C}} c_7^{2,4} = c_7^{2,3} b_5^{5,6} + b_{18}^{2,4} + b_2^{4,5} b_{18}^{5,6} + b_{18}^{2,3} b_7^{4,5} + b_{18}^{2,3} c_2^{2,3} b_2^{4,5} + b_{18}^{2,3} b_3^{2,3} b_2^{4,5} b_{18}^{5,6}, \tag{6.91}$$

$$\boxed{d_{\mathcal{C}} c_7^{3,5} = b_5^{5,6} c_7^{4,5},} \tag{6.92}$$

$$\boxed{d_{\mathcal{C}} c_7^{4,6} = c_7^{4,5} b_6^{3,4},} \tag{6.93}$$

$$d_{\mathcal{C}} c_7^{5,7} = b_6^{3,4} c_7^{6,7} + c_2^{2,3} + b_{18}^{3,5}, \tag{6.94}$$

$$d_{\mathcal{C}} c_7^{6,8} = c_7^{6,7} b_2^{4,5} + b_5^{5,6} b_{18}^{4,6} + b_5^{5,6} b_{18}^{4,5} b_{18}^{5,6}. \tag{6.95}$$

Among the above formulas, (6.94) implies that the generators

$$(c_7^{5,7}, c_2^{2,3}) \tag{6.96}$$

can be cancelled with each other.

For $n = m + 3$, based on the above computations we have

$$d_{\mathfrak{C}} c_7^{1,4} = b_3^{2,3} c_7^{2,4} + c_7^{1,2} b_2^{4,5} + c_7^{1,3} b_5^{5,6} + c_6^{4,5} + \dots, \tag{6.97}$$

where the ellipsis on the right-hand side above stands for some additional terms which do not play essential roles in our argument,

$$\boxed{d_{\mathfrak{C}} c_7^{2,5} = b_2^{4,5} c_7^{4,5} + b_{18}^{2,3},} \tag{6.98}$$

$$\boxed{d_{\mathfrak{C}} c_7^{3,6} = b_5^{5,6} c_7^{4,6} + c_7^{3,5} b_6^{3,4},} \tag{6.99}$$

$$\boxed{d_{\mathfrak{C}} c_7^{4,7} = c_7^{4,5} b_3^{2,3} + b_{18}^{4,5},} \tag{6.100}$$

$$d_{\mathfrak{C}} c_7^{5,8} = b_6^{3,4} c_7^{6,8} + c_7^{5,7} b_2^{4,5} + c_6^{4,5} + \dots. \tag{6.101}$$

From (6.97) we see that the generators

$$(c_7^{1,4}, c_6^{4,5}) \tag{6.102}$$

can be cancelled with each other.

For $n = m + 4$, based on the simplifications made above we deduce

$$\boxed{d_{\mathfrak{C}} c_7^{1,5} = b_3^{2,3} c_7^{2,5} + b_6^{3,4} c_7^{3,5} + b_7^{4,5} c_7^{4,5} + b_{18}^{1,3},} \tag{6.103}$$

$$d_{\mathfrak{C}} c_7^{2,6} = b_2^{4,5} c_7^{4,6} + c_7^{2,3} + c_7^{2,5} b_6^{3,4} + \dots, \tag{6.104}$$

$$d_{\mathfrak{C}} c_7^{3,7} = b_5^{5,6} c_7^{4,7} + c_7^{6,7} + c_7^{3,5} b_3^{2,3} + \dots, \tag{6.105}$$

$$\boxed{d_{\mathfrak{C}} c_7^{4,8} = c_7^{4,5} b_7^{4,5} + c_7^{4,6} b_5^{5,6} + c_7^{4,7} b_2^{4,5} + b_{18}^{4,6} + b_{18}^{4,5} b_{18}^{5,6}.} \tag{6.106}$$

(6.104) and (6.105) imply that the generators

$$(c_7^{2,6}, c_7^{2,3}), (c_7^{3,7}, c_7^{6,7}) \tag{6.107}$$

can be cancelled with each other.

For $n = m + 5$, we have

$$d_{\mathcal{C}} c_7^{1,6} = b_6^{3,4} c_7^{3,6} + b_7^{4,5} c_7^{4,6} + c_7^{1,3} + c_7^{1,5} b_6^{3,4} + \dots, \tag{6.108}$$

$$\boxed{d_{\mathcal{C}} c_7^{2,7} = b_2^{4,5} c_7^{4,7} + c_7^{2,5} b_3^{2,3} + b_{18}^{2,5} + b_{18}^{2,3} b_6^{3,4} c_7^{6,7} + b_{18}^{2,3} b_{18}^{3,5},} \tag{6.109}$$

$$d_{\mathcal{C}} c_7^{3,8} = b_5^{5,6} c_7^{4,8} + c_7^{6,8} + c_7^{3,5} b_7^{4,5} + c_7^{3,6} b_5^{5,6}. \tag{6.110}$$

From (6.108) and (6.110) we see that the generators

$$(c_7^{1,6}, c_7^{1,3}), (c_7^{3,8}, c_7^{6,8}) \tag{6.111}$$

can be cancelled in pair.

For $n = m + 6$, we get from the above that

$$d_{\mathcal{C}} c_7^{1,7} = b_3^{2,3} c_7^{2,7} + b_7^{4,5} c_7^{4,7} + c_7^{1,2} + \dots, \tag{6.112}$$

$$d_{\mathcal{C}} c_7^{2,8} = b_2^{4,5} c_7^{4,8} + c_7^{2,4} + \dots. \tag{6.113}$$

This enables us to cancel the generators

$$(c_7^{1,7}, c_7^{1,2}), (c_7^{2,8}, c_7^{2,4}) \tag{6.114}$$

with each other.

Finally, we have

$$d_{\mathcal{C}} c_7^{1,8} = b_7^{4,5} c_7^{4,8} + c_7^{5,8} + c_7^{1,5} b_7^{4,5} + \dots, \tag{6.115}$$

which implies that

$$(c_7^{1,8}, c_7^{5,8}) \tag{6.116}$$

can be cancelled with each other. We conclude from the above computations that the remaining generators of $C_7$ in the quotient dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$ are

$$c_7^{1,5}, c_7^{2,5}, c_7^{2,7}, c_7^{3,5}, c_7^{3,6}, c_7^{4,5}, c_7^{4,6}, c_7^{4,7}, c_7^{4,8}. \tag{6.117}$$

Since $B_{19} = 0$, by (6.33) applied to $j = 19$, we deduce that $A_3 = A_8$ in in $\mathcal{C}'(\Lambda_{2,2,2})$. This enables us to compute

$$\boxed{d_{\mathcal{C}} b_{11}^{4,5} = b_3^{2,3} b_2^{4,5} + b_{11}^{3,4} b_{10}^{5,6},} \tag{6.118}$$

together the differentials of the generators in $C_{11}$. The latter computation is in some sense symmetric to the computation of $d_{\mathcal{C}} c_7^{m,n}$. More explicitly, we have

$$d_{\mathcal{C}} c_{11}^{3,4} = c_8^{5,6}, \tag{6.119}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{4,5} = 0,} \tag{6.120}$$

$$d_{\mathcal{C}} c_{11}^{5,6} = c_9^{3,4}, \tag{6.121}$$

$$d_{\mathcal{C}} c_{11}^{7,8} = b_{18}^{5,6}, \tag{6.122}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{3,5} = b_{10}^{5,6} c_{11}^{4,5},} \tag{6.123}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{4,6} = c_{11}^{4,5} b_{11}^{3,4},} \tag{6.124}$$

$$d_{\mathcal{C}} c_{11}^{5,7} = b_6^{3,4} c_7^{6,7} + b_{18}^{3,5}, \tag{6.125}$$

$$d_{\mathcal{C}} c_{11}^{1,4} = b_3^{2,3} c_{11}^{2,4} + c_{11}^{1,2} b_2^{4,5} + c_{11}^{1,3} b_{10}^{5,6} + c_{10}^{4,5} + \dots, \tag{6.126}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{2,5} = b_2^{4,5} c_{11}^{4,5} + b_{18}^{2,3},} \tag{6.127}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{3,6} = b_{10}^{5,6} c_{11}^{4,6} + c_{11}^{3,5} b_{11}^{3,4},} \tag{6.128}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{4,7} = c_{11}^{4,5} b_3^{2,3} + b_{18}^{4,5},} \tag{6.129}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{1,5} = b_3^{2,3} c_{11}^{2,5} + b_{11}^{3,4} c_{11}^{3,5} + b_{12}^{4,5} c_{11}^{4,5} + b_{18}^{1,3},} \tag{6.130}$$

$$d_{\mathcal{C}} c_{11}^{2,6} = b_2^{4,5} c_{11}^{4,6} + c_{11}^{2,3} + c_{11}^{2,5} b_{11}^{3,4} + \dots, \tag{6.131}$$

$$d_{\mathcal{C}} c_{11}^{3,7} = b_{10}^{5,6} c_{11}^{4,7} + c_{11}^{6,7} + c_{11}^{3,5} b_3^{2,3} + \dots, \tag{6.132}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{4,8} = c_{11}^{4,5} b_{12}^{4,5} + c_{11}^{4,6} b_{10}^{5,6} + c_{11}^{4,7} b_2^{4,5} + b_{18}^{4,6},} \tag{6.133}$$

$$d_{\mathcal{C}} c_{11}^{1,6} = b_{11}^{3,4} c_{11}^{3,6} + b_{12}^{4,5} c_{11}^{4,6} + c_{11}^{1,3} + c_{11}^{1,5} b_{11}^{3,4} + \ldots, \tag{6.134}$$

$$\boxed{d_{\mathcal{C}} c_{11}^{2,7} = b_{2}^{4,5} c_{11}^{4,7} + c_{11}^{2,5} b_{3}^{2,3} + b_{18}^{2,5},} \tag{6.135}$$

$$d_{\mathcal{C}} c_{11}^{3,8} = b_{10}^{5,6} c_{11}^{4,8} + c_{11}^{6,8} + c_{11}^{3,5} b_{12}^{4,5} + c_{11}^{3,6} b_{10}^{5,6}, \tag{6.136}$$

$$d_{\mathcal{C}} c_{11}^{1,7} = b_{3}^{2,3} c_{11}^{2,7} + b_{12}^{4,5} c_{11}^{4,7} + c_{11}^{1,2} + \ldots, \tag{6.137}$$

$$d_{\mathcal{C}} c_{11}^{2,8} = b_{2}^{4,5} c_{11}^{4,8} + c_{11}^{2,4} + \ldots, \tag{6.138}$$

$$d_{\mathcal{C}} c_{11}^{1,8} = b_{12}^{4,5} c_{11}^{4,8} + c_{11}^{5,8} + c_{11}^{1,5} b_{12}^{4,5} + \ldots. \tag{6.139}$$

Arguing similarly as in the case of $C_7$, from (6.119), (6.121), (6.122), (6.125), (6.126), (6.131), (6.132), (6.134), (6.136), (6.137), (6.138), (6.139), we see that the following pairings of generators can be cancelled with each other:

$$\begin{gathered} (c_{11}^{3,4}, c_{8}^{5,6}), (c_{11}^{5,6}, c_{9}^{3,4}), (c_{11}^{7,8}, b_{18}^{5,6}), (c_{11}^{5,7}, b_{18}^{3,5}), \\ (c_{11}^{1,4}, c_{10}^{4,5}), (c_{11}^{2,6}, c_{11}^{2,3}), (c_{11}^{3,7}, c_{11}^{6,7}), (c_{11}^{1,6}, c_{11}^{1,3}), \\ (c_{11}^{3,8}, c_{11}^{6,8}), (c_{11}^{1,7}, c_{11}^{1,2}), (c_{11}^{2,8}, c_{11}^{2,4}), (c_{11}^{1,8}, c_{11}^{5,8}). \end{gathered} \tag{6.140}$$

We conclude that the remaining generators in $C_{11}$ are

$$c_{11}^{1,5}, c_{11}^{2,5}, c_{11}^{2,7}, c_{11}^{3,5}, c_{11}^{3,6}, c_{11}^{4,5}, c_{11}^{4,6}, c_{11}^{4,7}, c_{11}^{4,8}. \tag{6.141}$$

Combining (6.98) and (6.127) we see that one of the generators $c_7^{2,5}$ and $c_{11}^{2,5}$ can be cancelled with $b_{18}^{2,3}$. To be symmetric, we will denote the remaining generator in $\mathcal{C}'(\Lambda_{2,2,2})$ by $c_7^{2,5} + c_{11}^{2,5}$. It follows that

$$\boxed{d_{\mathcal{C}}(c_7^{2,5} + c_{11}^{2,5}) = b_2^{4,5} c_7^{4,5} + b_2^{4,5} c_{11}^{4,5}.} \tag{6.142}$$

Similarly, from (6.100) and (6.129), we obtain

$$\boxed{d_{\mathcal{C}}(c_7^{4,7} + c_{11}^{4,7}) = c_7^{4,5} b_3^{2,3} + c_{11}^{4,5} b_3^{2,3}.} \tag{6.143}$$

Meanwhile, $b_{18}^{4,5}$ is cancelled.

Also, (6.103) together with (6.130) implies that one can cancel one of $c_7^{1,5}$ and $c_{11}^{1,5}$ with $b_{18}^{1,3}$. As before, use $c_7^{1,5}+c_{11}^{1,5}$ to stand for the remaining generator in $\mathcal{C}'(\Lambda_{2,2,2})$, one has

$$\boxed{d_{\mathcal{C}}(c_7^{1,5}+c_{11}^{1,5}) = b_3^{2,3}(c_7^{2,5}+c_{11}^{2,5}) + b_6^{3,4}c_7^{3,5} + b_7^{4,5}c_7^{4,5} + b_{11}^{3,4}c_{11}^{4,5} + b_{12}^{4,5}c_{11}^{4,5}.} \tag{6.144}$$

We know from (6.122) that $b_{18}^{5,6}=0$ in the dg algebra $\mathcal{C}'(\Lambda_{2,2,2})$. Combining this fact with (6.106) and (6.133) we deduce that

$$\boxed{d_{\mathcal{C}}(c_7^{4,8}+c_{11}^{4,8}) = c_7^{4,5}b_7^{4,5} + c_7^{4,6}b_5^{5,6} + (c_7^{4,7}+c_{11}^{4,7})b_2^{4,5} + c_{11}^{4,5}b_{12}^{4,5} + c_{11}^{4,6}b_{10}^{5,6},} \tag{6.145}$$

and $b_{18}^{4,6}$ has been cancelled.

By (6.125), we know that

$$b_{18}^{3,5} = b_6^{3,4}c_7^{6,7} \tag{6.146}$$

holds in $\mathcal{C}'(\Lambda_{2,2,2})$. This combined with (6.109) and (6.135) implies that

$$\boxed{d_{\mathcal{C}}(c_7^{2,7}+c_{11}^{2,7}) = b_2^{4,5}(c_7^{4,7}+c_{11}^{4,7}) + (c_7^{2,5}+c_{11}^{2,5})b_3^{2,3},} \tag{6.147}$$

and $b_{18}^{2,5}$ is cancelled.

We have obtained all formulas of the differentials of the relevant generators in $\mathcal{C}'(\Lambda_{2,2,2})$. It remains to cancel all of the generators in $B_{18}$. Note that in the above, we have already cancelled $b_{18}^{5,6}$, $b_{18}^{3,5}$, $b_{18}^{2,3}$, $b_{18}^{4,5}$, $b_{18}^{1,3}$, $b_{18}^{4,6}$ and $b_{18}^{2,5}$. To cancel the remaining ones, apply (6.35) to the case when $j=18$. Using the fact that $A_8=A_3$ in $\mathcal{C}'(\Lambda_{2,2,2})$, we have

$$d_{\mathcal{C}}b_{18}^{m,n} = \sum_{m<k<n} a_3^{\sigma_4(m),\sigma_4(k)}b_{18}^{k,n} + \sum_{m<k<n} b_{18}^{m,k}a_3^{\sigma(k),\sigma(n)}. \tag{6.148}$$

Since $a_3^{1,2}=a_3^{3,5}=a_3^{4,6}=1$, we deduce from (6.148) that the generators

$$(b_{18}^{1,4},b_{18}^{3,4}),(b_{18}^{1,6},b_{18}^{3,6}),(b_{18}^{1,5},b_{18}^{1,2}),(b_{18}^{2,6},b_{18}^{2,4}) \tag{6.149}$$

can be cancelled with each other.

The final step is to compute the gradings of the generators in $\mathcal{C}'(\Lambda_{2,2,2})$. To do this, we equip the sheets in the Legendrian surface $\Lambda_{2,2,2}$ with the Maslov potential $\mu_{2,2,2} : \Lambda_{2,2,2} \to \mathbb{Z}$ as specified in Figure 4.5. It is then straightforward to check that the gradings of the remaining generators in $\mathcal{C}'(\Lambda_{2,2,2})$ are exactly as in (6.48). $\square$

## 6.3 Calabi-Yau completions as wrapped Fukaya categories

In this section, we complete the proof of Theorem 1.5.1. We begin by recalling the definition of the Chekanov-Eliashberg dg algebra $CE^*(\Lambda)$ over $\mathbb{K} = \mathbb{Z}/2$ of a Legendrian surface $\Lambda \subset J^1(\mathbb{R}^2)$. More details can be found in [34]. As in Section 5.1, assume that $\Lambda$ has vanishing Maslov class so that $CE^*(\Lambda)$ is $\mathbb{Z}$-graded.

Denote by $\mathcal{R}$ the set of transverse double points in the Lagrangian projection $p_{xy}(\Lambda)$. Without loss of generality, we can assume that $\Lambda$ is *chord generic*, namely $\mathcal{R}$ is a finite set. As a graded $\mathbb{K}$-algebra,

$$CE^*(\Lambda) := \bigoplus_{i=0}^{\infty} \mathbb{K}\langle\mathcal{R}\rangle^{\otimes_{\mathbb{K}} i}. \tag{6.150}$$

As explained in Section 5.1, when $\Lambda = \bigsqcup \Lambda_v$ is a Legendrian link, $CE^*(\Lambda)$ can be regarded as a dg algebra over $\Bbbk$, such that $e_w \mathcal{R} e_v$ consists of the Reeb chords from $\Lambda_w$ to $\Lambda_v$. With respect to the differential $\partial$ defined below, this endows $CE^*(\Lambda)$ with the structure of a dg algebra over $\Bbbk$.

For any transverse double point $c \in \mathcal{R}$, its pre-image consists of two points, $c_+$ and $c_-$, where $z(c_+) > z(c_-)$. By slight abuse of notations, their images under the front projection $p_{xz}$ will still be denoted by $c_+$ and $c_-$. By assumption, they project to the same point $x_c \in \mathbb{R}^2$. Let $f_+$ and $f_-$ be the local defining functions of the sheets of $\Lambda$ which contain the points $c_+$ and $c_-$ respectively. $x_c$ is a non-degenerate critical point of the local difference function $f := f_+ - f_-$, so it has an associated Morse index $ind(x_c)$. Choose a path $\gamma$ from $c_+$ to $c_-$, which is transverse to the singular set of $p_{xz}(\Lambda)$. It follows that $\gamma$ intersects the cusp edges of $p_{xz}(\Lambda)$ in a finite number of points. Denote by $d(\gamma)$ the number of times that $\gamma$ crosses from the upper sheet to the lower sheet, and by $u(\gamma)$ the number of times that $\gamma$ crosses from the lower sheet to the upper sheet, the grading of $c$ in the dg algebra $CE^*(\Lambda)$ is defined to be

$$|c| = u(\gamma) + 1 - ind(x_c) - d(\gamma). \tag{6.151}$$

For generators $a; b_1, \ldots, b_l \in \mathcal{R}$, we can define a moduli space $\mathcal{M}_\Lambda(a; b_1, \ldots, b_l)$ which parametrizes holomorphic maps

$$u : (\Delta_{l+1}, \partial\Delta_{l+1}) \to (\mathbb{C}^2, p_{xy}(\Lambda)), \tag{6.152}$$

where $\Delta_{l+1}$ is an $(l+1)$-punctured disc with punctures labelled counterclockwisely by $q_0, \ldots, q_l$ on the boundary. As in the case of a Legendrian link in $(\mathbb{R}^3, \xi_{std})$, we need to introduce the *Reeb sign* for the punctures. For a Reeb chord $c \in \mathcal{R}$, pick small neighborhoods $S_\pm \subset \Lambda$ of $c_\pm$ that are mapped injectively by $p_{xy}$ into $\mathbb{C}^2$. If $u(q_i) = c$, we say that $q_i$ has *positive* (resp. *negative*) Reeb sign if $u$ maps points clockwise of $p_i$ on $\partial\Delta_{l+1}$ to the lower (resp. upper) sheet of $p_{xy}(\Lambda)$, and points counterclockwise of $p_i$ on $\partial\Delta_{l+1}$ to the upper (resp. lower) sheet of $p_{xy}(\Lambda)$. See Figure 6.6. Furthermore, $u$ is required to satisfy the following boundary and asymptotic conditions:

- $u$ maps the boundary components of the punctured disc $\Delta_{l+1}$ to $p_{xy}(\Lambda) \subset \mathbb{C}^2$;
- $u(q_0) = a$ has positive Reeb sign and $u(q_i) = b_i$ has negative Reeb sign for $i = 1, \ldots, l$.

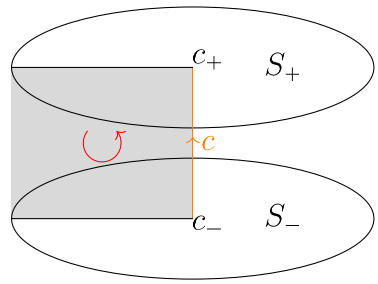


Figure 6.6: A positive puncture lifted to $\mathbb{R}^5$, where the shaded region is the image of a holomorphic disc

The differential $\partial$ in $CE^*(\Lambda)$ is defined as

$$\partial a = \sum(\#_2\mathcal{M}_\Lambda(a; b_1, \ldots, b_l))b_1 \ldots b_l, \tag{6.153}$$

where the sum on the right-hand side above is taken over all words $b_1 \ldots b_l$ of Reeb chords for which $\dim(\mathcal{M}_\Lambda(a; b_1, \ldots, b_l)) = 0$ and $\#_2$ denotes the mod 2 count of the rigid elements in $\mathcal{M}_\Lambda(a; b_1, \ldots, b_l)$. In order to define $CE^*(\Lambda)$ over an arbitrary field $\mathbb{K}$, we need to take into consideration the orientations of the Morse flow trees which correspond to rigid holomorphic discs in $\mathcal{M}(a; b_1, \ldots, b_l)$, see Appendix A.

Recall that it is proved by Rutherford-Sullivan in [86] that the cellular dg algebra $\mathcal{C}(\Lambda)$ of a Legendrian surface $\Lambda \subset J^1(\mathbb{R}^2)$ is quasi-isomorphic to its Chekanov-Eliashberg algebra $CE^*(\Lambda)$ defined over $\mathbb{K} = \mathbb{Z}/2$. In particular, we have a quasi-isomorphism

$$\mathcal{C}(\Lambda_{p,q,r}) \cong CE^*(\Lambda_{p,q,r}). \tag{6.154}$$

Note that this quasi-isomorphism preserves the $\Bbbk$-bimodule structures on both sides.

It remains to identify the cellular dg algebra $\mathcal{C}(\Lambda_{p,q,r})$ with the Ginzburg algebra $\mathcal{G}_{p,q,r}$. This follows essentially from our computations in Sections 6.1 and 6.2.

**Proposition 6.3.1.** *Let $p, q$ and $r$ be integers satisfying $p \geqslant 2, q \geqslant 2, r \geqslant 2$. We have a quasi-isomorphism*

$$\mathcal{C}(\Lambda_{p,q,r}) \cong \mathcal{G}_{p,q,r} \tag{6.155}$$

*between the cellular dg algebra and the Ginzburg dg algebra defined over $\mathbb{K} = \mathbb{Z}/2$.*

*Proof.* By definition, the Ginzburg algebra $\mathcal{G}_{p,q,r}$ is the semi-free dg algebra generated by the arrows (which have degree 0)

$$a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3, x_i, y_j, z_k; \tag{6.156}$$

the reversed arrows (which have degree -1)

$$a_1^*, a_2^*, b_1^*, b_2^*, b_3^*, c_1^*, c_2^*, c_3^*, x_i^*, y_j^*, z_k^*; \tag{6.157}$$

together with the loops (which have degree -2)

$$z_A, z_B, z_{P_1}, z_{Q_1}, z_{R_1}, z_{P_{i+1}}, z_{Q_{j+1}}, z_{R_{k+1}}, \tag{6.158}$$

The differential $d : \mathcal{G}_{p,q,r} \to \mathcal{G}_{p,q,r}[1]$ is determined by its action on the generators listed above. More explicitly,

$$da_1 = da_2 = db_1 = db_2 = db_3 = dc_1 = dc_2 = dc_3 = dx_i = dy_j = dz_k = 0, \tag{6.159}$$

$$dx_1 = \cdots = dx_{p-1} = dy_1 = \cdots = dy_{q-1} = dz_1 = \cdots = dz_{r-1} = 0, \tag{6.160}$$

$$da_1^* = b_2c_2 + b_3c_3, da_2^* = b_1c_1 + b_3c_3, \tag{6.161}$$

$$db_1^* = c_1a_2, db_2^* = c_2a_1, db_3^* = c_3a_1 + c_3a_2, \tag{6.162}$$

$$dc_1^* = a_2b_1, dc_2^* = a_1b_2, dc_3^* = a_1b_3 + a_2b_3, \tag{6.163}$$

$$dx_1^* = \cdots = dx_{p-1}^* = dy_1^* = \cdots = dy_{q-1}^* = dz_1^* = \cdots = dz_{r-1}^* = 0, \tag{6.164}$$

$$dz_A = a_1a_1^* + a_2a_2^* + c_1^*c_1 + c_2^*c_2 + c_3^*c_3, \tag{6.165}$$

$$dz_B = a_1^* a_1 + a_2^* a_2 + b_1 b_1^* + b_2 b_2^* + b_3 b_3^*, \tag{6.166}$$

$$dz_{P_1} = c_1 c_1^* + b_1^* b_1 + x_1 x_1^*, dz_{Q_1} = c_2 c_2^* + b_2 b_2^* + y_1 y_1^*, dz_{R_1} = c_3 c_3^* + b_3^* b_3 + z_1 z_1^*, \tag{6.167}$$

$$dz_{P_i} = x_{i-1}^* x_{i-1} + x_i x_i^*, dz_{Q_j} = y_{j-1}^* y_{j-1} + y_j y_j^*, dz_{R_k} = z_{k-1}^* z_{k-1} + z_k z_k^*, \tag{6.168}$$

where $2 \leqslant i \leqslant p-2$, $2 \leqslant j \leqslant q-2$, $2 \leqslant k \leqslant r-2$, and finally

$$dz_{P_{p-1}} = x_{p-2}^* x_{p-2}, dz_{Q_{q-1}} = y_{q-2}^* y_{q-2}, dz_{R_{r-1}} = z_{r-2}^* z_{r-2}. \tag{6.169}$$

On the other hand, by Proposition 6.2.1, it is clear that the map $\Phi_{2,2,2} : \mathcal{C}(\Lambda_{2,2,2}) \to \mathcal{G}_{2,2,2}$ defined by

$$\begin{aligned}
&c_{12}^{4,5} \mapsto a_1, c_7^{4,5} \mapsto a_2, b_6^{3,4} \mapsto b_1, b_{11}^{3,4} \mapsto b_2, b_3^{2,3} \mapsto b_3, b_5^{5,6} \mapsto c_1, b_{10}^{5,6} \mapsto c_2, b_2^{4,5} \mapsto c_3;\\
&b_{12}^{4,5} \mapsto a_1^*, b_7^{4,5} \mapsto a_2^*, c_7^{3,5} \mapsto b_1^*, c_{11}^{3,5} \mapsto b_2^*,\\
&c_7^{2,5} + c_{11}^{2,5} \mapsto b_3^*, c_7^{4,6} \mapsto c_1^*, c_{11}^{4,6} \mapsto c_2^*, c_7^{4,7} + c_{11}^{4,7} \mapsto c_3^*;\\
&c_7^{1,5} + c_{11}^{1,5} \mapsto z_A, c_7^{4,8} + c_{11}^{4,8} \mapsto z_B, c_7^{3,6} \mapsto z_P, c_{11}^{3,6} \mapsto z_Q, c_7^{2,7} + c_{11}^{2,7} \mapsto z_R
\end{aligned} \tag{6.170}$$

defines an identification between the generators of the dg algebras, and it is straightforward to check that this map is compatible with the differentials. In particular, $\Phi_{2,2,2}$ is a quasi-isomorphism.

After replacing the components $\Lambda_P, \Lambda_Q$ and $\Lambda_R$ of $\Lambda_{2,2,2}$ with the corresponding $A_{p-1}$, $A_{q-1}$ and $A_{r-1}$ chains of unknots $\{\Lambda_{P_i}\}$, $\{\Lambda_{Q_j}\}$ and $\{\Lambda_{R_k}\}$, one can arrange so that the base projection $p_x(\Lambda_{p,q,r})$ is as shown in Figure 6.4, where the additional cells in the cellular decomposition of $p_x(\Lambda_{p,q,r})$ are coloured in blue. Note that we have arranged, by using a non-generic front projection of $\Lambda_{p,q,r}$, that the base projections of the crossing arcs in the $A_{p-1}$ and $A_{q-1}$ chains of unknots $\{\Lambda_{P_i}\}$ and $\{\Lambda_{Q_j}\}$ coincide precisely with the 1-cells $e_1^1$ and $e_8^1$, while the crossing arcs of the $A_{r-1}$-chain of unknots lie over $e_{14}^1$. The base projections of the cusp edges of $p_{x,z}(\Lambda_{P_i})$, $p_{x,z}(\Lambda_{Q_j})$

and $p_{x,z}(\Lambda_{R_k})$ are given respectively by the 1-cells $e^1_{15}$, $e^1_{18}$ and the largest circle, so there are no newly created 1-cells for these additional cusp edges in the Legendrian front of $\Lambda_{p,q,r}$. This arrangement of the Legendrian front of $\Lambda_{p,q,r}$ is justified by our discussions in Section 5.2. Note that since we have to add the additional cells labelled by $A_{13}, A_{14}, B_{22}$ to divide the 2-cell labelled by $C_3$ into a polygon, the original 1-cell $e^1_{20}$ in the cellular decomposition of $p_x(\Lambda_{2,2,2})$ is divided into two 1-cells, say $e^1_{20,+}$ and $e^1_{20,-}$. However, since we can cancel the generators in one of the matrices $B_{20,+}$ and $B_{20,-}$ with the generators in $A_{13}$, it causes no additional complexity in our computations. Using the same cancellation arguments as in the proof of Proposition 6.2.1, it is not hard to see that passing from $\mathcal{C}'(\Lambda_{2,2,2})$ to $\mathcal{C}'(\Lambda_{p,q,r})$ adds new generators in the matrices $B_7, C_7, B_{12}, C_{11}, B_{22}$ and $C_{12}$. All the other new generators involved in the original definition of the cellular dg algebra $\mathcal{C}(\Lambda_{p,q,r})$, including those coming from the newly added cells in the base projection of $\Lambda_{p,q,r}$, can be cancelled out.

For the new generators created by the parallel copies of $\Lambda_P$, $\Lambda_Q$ and $\Lambda_R$, namely $\Lambda_{P_i}$, $\Lambda_{Q_j}$ and $\Lambda_{R_k}$ for $i, j, k \geqslant 2$, we can apply exactly the same cancellation procedure as in the proof of Proposition 6.1.1. The upshot is that up to quasi-isomorphism, the additional generators in $\mathcal{C}'(\Lambda_{p,q,r})$ can be identified to be

$$x_i, x_i^*, y_j, y_j^*, z_k, z_k^*, z_{P_2}, \dots, z_{P_{p-1}}, \tag{6.171}$$

where $1 \leqslant i \leqslant p-1$, $1 \leqslant j \leqslant q-1$, $1 \leqslant k \leqslant r-1$, with the differentials given exactly as in (6.168) and (6.169) above. Furthermore, in addition to the new generators in (6.171), there are additional terms $x_1x_1^*, y_1y_1^*$ and $z_1z_1^*$ appearing in the differentials of $z_{P_1}, z_{Q_1}$ and $z_{R_1}$ respectively.

Finally, notice that the Maslov potential $\mu_{2,2,2}$ on the Legendrian link $\Lambda_{2,2,2}$ extends naturally to a Maslov potential $\mu_{p,q,r} : \Lambda_{p,q,r} \to \mathbb{Z}$, by equipping the additional unknots in $\Lambda_{p,q,r}$ with a Maslov potential as in (6.22). This shows that the grading on $\mathcal{C}(\Lambda_{p,q,r})$ matches with that on $\mathcal{G}_{p,q,r}$, which completes the proof. $\square$

By (6.154) we get a quasi-isomorphism

$$CE^*(\Lambda_{p,q,r}) \cong \Pi_3(\mathcal{A}_{p,q,r}) \tag{6.172}$$

between the Chekanov-Eliashberg algebra and the 3-Calabi-Yau completion of the directed $A_\infty$-algebra $\mathcal{A}_{p,q,r}$. With our definitions, it is straightforward to see that this quasi-isomorphism is compatible with the $\Bbbk$-bimodule structures on both sides. This proves Theorem 1.5.1 for $\mathbb{K} = \mathbb{Z}/2$. The general case is proved by combining the computations of the signs of the relevant Morse flow trees, which is carried out in Section A.2.

The notion of a Calabi-Yau completion has its relative counterpart, namely the *relative Calabi-Yau completions* introduced by Yeung [116]. For related results which identify certain (partially) wrapped Fukaya categories with relative Calabi-Yau completions, see [41] and [116].

## 6.4 Degenerate triples

For this section, $\mathbb{K}$ can be any field unless otherwise specified. Let the polynomial $t_{p,q,r}(x,y,z)$ be as in (1.20), it gives rise to a symplectic Landau-Ginzburg model $(\mathbb{C}^3, t_{p,q,r})$. Without loss of generality, we assume that $p \geqslant p \geqslant r \geqslant 0$. It turned out that these Landau-Ginzburg models have 1-dimensional mirrors.

When $r \geqslant 2$, the Morsification $\tilde{t}_{p,q,r}$ defines a Lefschetz fibration on $\mathbb{C}^3$, and the mirror of $(\mathbb{C}^3, \tilde{t}_{p,q,r})$ is the weighted projective line $\mathbb{P}^1_{p,q,r}$. The Fukaya categories $\mathcal{F}(M_{p,q,r})$ and $\mathcal{W}(M_{p,q,r})$ of the corresponding Wesintein manifold $M_{p,q,r}$ have been studied in the previous chapters. As we have proved, they are $A_\infty$-Koszul dual to each other as $\mathbb{Z}$-graded $A_\infty$-categories.

When $r = 1$, the situation is a simplification of the previous case. The mirror of $(\mathbb{C}^3, \tilde{t}_{p,q,1})$ is $\mathbb{P}^1_{p,q}$. When $q \geqslant 2$, the Fukaya categories of $M_{p,q,1}$ are described by the quiver $Q_{p,q,1}$

$$\begin{tikzcd}
 & & \bullet_{P_1} \arrow[r, "x_1"] \arrow[dll, bend right, "c_1"'] & \bullet_{P_2} \arrow[r, "x_2"] & \bullet \cdots \bullet \arrow[r, "x_{p-2}"] & \bullet_{P_{p-1}} \\
\bullet_A \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] & \bullet_B \arrow[ur, "b_1"] \arrow[dr, "b_2"'] & & & & \\
 & & \bullet_{Q_1} \arrow[r, "y_1"] \arrow[ull, bend left, "c_2"] & \bullet_{Q_2} \arrow[r, "y_2"] & \bullet \cdots \bullet \arrow[r, "y_{q-2}"] & \bullet_{Q_{q-1}}
\end{tikzcd} \tag{6.173}$$

with potential

$$w_{p,q,1} = a_1 b_2 c_2 + a_2 b_1 c_1. \tag{6.174}$$

In the simplest case when $p = q = r = 1$, $Q_{1,1,1}$ is just the Kronecker quiver, and the potential $w_{1,1,1} = 0$. For $M_{p,q,1}$, one can still find a Lefschetz fibration $\pi_{p,q,1} : M_{p,q,1} \to \mathbb{C}$ (although the construction of $\pi_{p,q,1}$ does not follow from the general method described in Section 4.1) and use the Casals-Murphy recipe to draw its Legendrian front. It turns out that one can cancel all the subcritical handles in the original frontal description of $M_{p,q,1}$ to get a Legendrian surface $\Lambda_{p,q,1} \subset (\mathbb{R}^5, \xi_{std})$ and the Chekanov-Eliashberg algebra $CE^*(\Lambda_{p,q,1})$ can still be computed through its cellular model $\mathcal{C}(\Lambda_{p,q,1})$. The proof of Koszul duality between the endomorphism algebras of $\mathcal{F}(M_{p,q,1})$ and $\mathcal{W}(M_{p,q,1})$ is completely analogous to the previous case.

When $r = 0$ and $q \geqslant 1$, the map $t_{p,q,0} : \mathbb{C}^3 \to \mathbb{C}$ cannot be Morsified to produce a Lefschetz fibration, and the mirror of $(\mathbb{C}^3, t_{p,q,0})$ is $\mathbb{A}^1_{p,q} := \mathbb{P}^1_{p,q} \backslash \{\infty\}$. When $p = q = 1$, the map $t_{1,1,0} : \mathbb{C}^3 \to \mathbb{C}$ does not have any critical point. Instead, there is a special fiber over the origin, which is topologically different from all the other fibers. Correspondingly, its mirror $\mathbb{A}^1$ is Floer theoretically trivial over $\mathbb{K} = \mathbb{C}$. The Landau-Ginzburg model $(\mathbb{C}^3, t_{1,1,0})$ has already been studied in Example 2.4 of [6], where it is equivalently interpreted as the 1-dimensional Landau-Ginzburg model $(\mathbb{C}^*, x)$. In fact, let $L_\infty \subset \mathbb{C}^*$ be a properly embedded arc which connects $+\infty$ to itself by passing around the origin, then

$$\mathcal{A}_{1,1,0} \cong CF^*(L_\infty, L_\infty) \cong H^*(S^1; \mathbb{K}), \tag{6.175}$$

which shows that $L_\infty$ is mirror to the skyscraper sheaf at the origin of $\mathbb{A}^1$. $\mathcal{A}_{1,1,0}$ should be regarded as the endomorphism algebra of the *infinitesimally wrapped Fukaya category* (cf. [82]) $\mathcal{A}(t_{1,1,0})$. Geometrically, it still makes sense to consider the trivial extension algebra $\mathcal{A}_{1,1,0} \oplus \mathcal{A}_{1,1,0}^\vee[-3]$, since one can construct an unobstructed (but non-exact) Lagrangian submanifold $L_\sigma \subset M$, which is diffeomorphic to $S^1 \times S^2$ and has trivial Maslov class. Now the trivial extension $\mathcal{A}_{1,1,0} \oplus \mathcal{A}_{1,1,0}^\vee[-3]$ can be regarded as the endomorphism algebra $CF^*(L_\sigma, L_\sigma)$ of $L_\sigma$ in the Fukaya category $\mathcal{F}(M_{1,1,0})$ (which extends the usual definition by allowing all the closed Lagrangian submanifolds $L \subset M_{1,1,0}$ with trivial obstructions $\mathfrak{m}_0(L) = 0$ as its objects). This suggests the existence of a generalization of the suspension construction of Lefschetz fibrations explained in Section 3.1 to more general symplectic Landau-Ginzburg models.

From Section 4.1 of [22], we see that $M_{1,1,0}$ is obtained by attaching a Weinstein 3-handle to $D^6$ along the Legendrian surface $\Lambda_{1,1,0}$ depicted in Figure 6.7. Although in [22], the Legendrian front $\Lambda_{1,1,0}$ of $M_{1,1,0}$ is obtained using a different Lefschetz fibration, one will end up with the same front by starting from our general framework in Section 4.3, and cancelling the 2-handle $R$ with the 3-handle corresponding to the Legendrian sphere $\Lambda_A$, see Figure 4.3. In particular, this shows that for a general

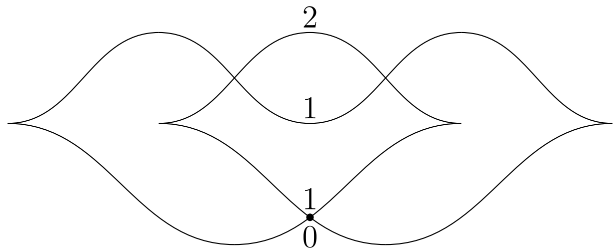


Figure 6.7: Legendrian front of $\Lambda_{1,1,0}$, which is the $S^1$-symmetric rotation of a right-handed trefoil knot with respect to the vertical axis of symmetry. The numbers above strands are values taken by the Maslov potential $\mu_{1,1,0} : \Lambda_{1,1,0} \to \mathbb{Z}$.

Weinstein manifold $M_{p,q,0}$, one can cancel all the subcritical handles in its frontal description, and the endomorphism algebra of its wrapped Fukaya category $\mathcal{W}(M_{p,q,0})$ is quasi-isomorphic to the Chekanov-Eliashberg dg algebra of a Legendrian surface $\Lambda_{p,q,0} \subset (\mathbb{R}^5, \xi_{std})$, which is of course computable in terms of the cellular dg algebra $\mathcal{C}(\Lambda_{p,q,0})$. Concretely, $\Lambda_{p,q,0}$ is a link of Legendrian 2-spheres obtained by attaching standard unknots to the surface $\Lambda_{1,1,0}$.

Since the Legendrian surface $\Lambda_{1,1,0} \subset J^1(\mathbb{R}^2)$ can be constructed by spinning the Legendrian front of a right-handed Legendrian trefoil knot in $(\mathbb{R}^3, \xi_{std})$ along an axis which passes through one of the crossing points, its front projection contains a cone singularity, which is indicated by the thick dot in Figure 6.7. Its Chekanov-Eliashberg dg algebra $CE^*(\Lambda_{1,1,0})$ can be computed by applying Proposition 5.4.1, and there is a quasi-isomorphism

$$CE^*(\Lambda_{1,1,0}) \cong \mathbb{Z}/2[x_1, x_2], |x_1| = 1, |x_2| = -2 \tag{6.176}$$

over $\mathbb{Z}/2$, see Appendix B for details. In particular, this implies that $CE^*(\Lambda_{1,1,0})$ is not Koszul dual to $CF^*(L_\sigma, L_\sigma)$ over $\mathbb{K} = \mathbb{Z}/2$.

When $q = r = 0$, and $p \geqslant 1$, the Landau-Ginzburg model $(\mathbb{C}^3, t_{p,0,0})$ is mirror to $\mathbb{K}_p^\times := \mathbb{P}_p^1 \backslash \{0, \infty\}$. In the simplest case when $p = 1$, the map $t_{1,0,0} : \mathbb{C}^3 \to \mathbb{C}$ defines a Morse-Bott fibration with critical locus isomorphic to $\mathbb{C}^*$. The main difficulty in answering whether the Fukaya categories $\mathcal{F}(M_{p,0,0})$ and $\mathcal{W}(M_{p,0,0})$ are Koszul dual comes from the fact that the Legendrian frontal description of $M_{p,0,0}$ involves a 2-handle which cannot be cancelled with any of the critical handles, therefore the calculation of the wrapped Fukaya category $\mathcal{W}(M_{p,0,0})$ requires a generalization of the work of Rutherford-Sullivan [85, 86] for Legendrian surfaces in the contact connected sums $\#_n S^2 \times S^3$. In dimension 4, the corresponding generalization has been obtained by Ekholm-Ng [37].

The Landau-Ginzburg model $(\mathbb{C}^3, t_{0,0,0})$ has been studied prominently in the literature, see for example [6, 81]. Its mirror is given by the pair-of-pants $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. The Fukaya category of $(\mathbb{C}^3, t_{0,0,0})$ has been calculated by Nadler in [81] in terms of microlocal sheaves, which turns out to be quasi-equivalent to $\mathcal{A}(t_{0,0,0}) := Coh_{tor}(\mathbb{P}^1 \backslash \{0, 1, \infty\})$, the bounded dg category of finitely-generated torsion complexes on $\mathbb{P}^1 \backslash \{0, 1, \infty\}$. The full $A_\infty$-subcategory $\mathcal{V}(M_{0,0,0}) \subset \mathcal{F}(M_{0,0,0})$ which is relevant for Koszul duality should be the trivial extension $\mathcal{A}(t_{0,0,0}) \oplus \mathcal{A}(t_{0,0,0})^\vee[-3]$. However, as in the case of $M_{p,0,0}$, the Legendrian front of $M_{0,0,0}$ necessarily involves 2-handles.

# Chapter 7

# Parametrized Floer theory

From now on, fix $\mathbb{K}$ to be a field of characteristic zero. In order to study the exact Calabi-Yau structures on wrapped Fukaya categories from the perspective of symplectic geometry, we need some background on parametrized Floer theory. As the general geometric set up, we assume that $(M, \theta_M)$ is a $2n$-dimensional Liouville manifold. In order to have $\mathbb{Z}$-gradings on various Floer cochain complexes, we assume that $c_1(M) = 0$, and will in fact fix the choice of a trivialization of the canonical bundle $K_M$.

## 7.1 Equivariant symplectic cohomology

We recall the definition of $S^1$-equivariant symplectic cohomology $SH^*_{S^1}(M)$, whose construction is sketched in [90] and later carried out in detail by Bourgeois-Oancea in [19]. We will actually follow closely the general framework of Ganatra [43], which has the advantage of being coordinate-free. The construction of $SH^*_{S^1}(M)$ is more involved than its non-equivariant version $SH^*(M)$ in the sense that besides the Floer differential $\delta_0 = d$, there is now a sequence of higher order corrections $\delta_k$, $k \geqslant 1$, whose definitions make use of parametrized moduli spaces. There is therefore an additional set of moduli spaces, namely the moduli spaces of domains defining $\{\delta_k\}_{k\geqslant 1}$, that enters into our discussions below.

**Definition 7.1.1** ([43], Definition 11)**.** *A $k$-point angle-decorated cylinder consists of a cylinder $Z = \mathbb{R} \times S^1$, together with a collection of auxiliary marked points $p_1, \cdots, p_k \in Z$, such that their $s \in \mathbb{R}$ coordinates $(p_i)_s, 1 \leqslant i \leqslant k$ satisfy*

$$(p_1)_s \leqslant \cdots \leqslant (p_k)_s. \tag{7.1}$$

*We call these coordinates the heights of the marked points, and denote them by*

$$h_i := (p_i)_s, i = 1, \cdots, k. \tag{7.2}$$

*Similarly, the $t \in S^1$ coordinates $(p_i)_t, 1 \leqslant i \leqslant k$ of the marked points are called angles, and we introduce the notations*

$$\theta_i := (p_i)_t, i = 1, \cdots, k. \tag{7.3}$$

Denote by $\mathcal{M}_k$ the moduli space of $k$-point angle-decorated cylinders, modulo translation in the $s$-direction. Given a marked cylinder $(Z, p_1, \cdots, p_k)$ representing an element of $\mathcal{M}_k$. For a fixed constant $\xi > 0$, define positive and negative cylindrical ends

$$\varepsilon^+ : [0, \infty) \times S^1 \to Z \text{ and } \varepsilon^- : (-\infty, 0] \times S^1 \to Z \tag{7.4}$$

by

$$\varepsilon^+(s,t) = (s + h_k + \xi, t) \text{ and } \varepsilon^-(s,t) = (s + h_1 - \xi, t + \theta_1) \tag{7.5}$$

respectively.

$\mathcal{M}_k$ can be compactified to a manifold with corners $\overline{\mathcal{M}}_k$ by adding broken $k$-point angle-decorated cylinders, by which we mean

$$\bigsqcup_s \bigsqcup_{j_1,\cdots,j_s; j_i>0, \sum j_i = k} \mathcal{M}_{j_1} \times \cdots \times \mathcal{M}_{j_s}. \tag{7.6}$$

However, there are additional boundary strata corresponding to the cases when one or several of the auxiliary marked points share the same height. In particular, the codimension 1 boundary of $\overline{\mathcal{M}}_k$ is covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_{k-j} \times \overline{\mathcal{M}}_j \hookrightarrow \partial\overline{\mathcal{M}}_k, 0 < j < k; \tag{7.7}$$

$$\overline{\mathcal{M}}_k^{i,i+1} \hookrightarrow \partial\overline{\mathcal{M}}_k, \tag{7.8}$$

where $\overline{\mathcal{M}}_k^{i,i+1}$ is the compactification of the locus where $h_i = h_{i+1}$. On $\mathcal{M}_k^{i,i+1}$ there is a forgetful map

$$\pi_i : \mathcal{M}_k^{i,i+1} \to \mathcal{M}_{k-1} \tag{7.9}$$

which remembers only the first one of the angles of the interior marked points with coincident heights. Since $\pi_i$ is compatible with the choices of cylindrical ends $\varepsilon^{\pm}$ specified by (7.4), it extends to a map $\bar{\pi}_i : \overline{\mathcal{M}}_k^{i,i+1} \to \overline{\mathcal{M}}_{k-1}$ defined on the compactifications.

In order to write down the Floer equations, we need to introduce Floer data on the domain cylinders. To do this, one needs to specify the sets of Hamiltonian functions and almost complex structures to work with. We say that a time-dependent Hamiltonian $H_t : S^1 \times M \to \mathbb{R}$ is *admissible* if $H_t = H + F_t$ is the sum of an autonomous Hamiltonian $H : M \to \mathbb{R}$ which is quadratic at infinity, namely

$$H(r, y) = r^2 \tag{7.10}$$

on the cylindrical end $[r_0, \infty) \times \partial\overline{M}$, where $r \in (1, \infty)$ is the radial coordinate and $r_0 \gg 1$, and a time-dependent perturbation $F_t : S^1 \times M \to \mathbb{R}$. We require that on $M \backslash \overline{M}$, we have that for any $r_1 \gg 0$, there exists an $r > r_1$ such that $F_t$ vanishes in a neighborhood of the hypersurface $\{r\} \times \partial\overline{M} \subset M$. For instance, one can take $F_t$ to be a function supported near non-constant orbits of $X_H$, where it is modelled on a Morse function on $S^1$.

Denote by $\mathcal{H}(M)$ the set of admissible Hamiltonians $H_t$ on $M$ such that all 1-periodic orbits of the Hamiltonian vector field $X_{H_t}$ are non-degenerate, and write $\mathcal{O}_M$ for the set of 1-periodic orbits of $X_{H_t}$. For an orbit $y \in \mathcal{O}_M$, we define its degree to be

$$\deg(y) := n - CZ(y), \tag{7.11}$$

where $CZ(y)$ is the Conley-Zehnder index of $y$. With these data one can define a $\mathbb{Z}$-graded Floer cochain complex of $H_t$, which is called the *symplectic cochain complex*, and will be denoted by $SC^*(M)$, whose degree $i$ piece is given by

$$SC^i(M) := \bigoplus_{y \in \mathcal{O}_M, \deg(x) = i} |o_y|_{\mathbb{K}}, \tag{7.12}$$

where $|o_y|_{\mathbb{K}}$ is the $\mathbb{K}$-normalization of the orientation line $o_y$ defined via index theory.

**Remark 7.1.1.** *For convenience, we will often choose generators of the orientation lines $o_y$ associated to each Hamiltonian orbit and denote them by $y$ in a slight abuse of notation. The same convention applies to orientation lines associated Hamiltonian chords when dealing with open-string invariants.*

Let $\mathcal{J}(M)$ be the set of $d\theta_M$-compatible $t$-dependent alsmost complex structures on $M$ which are *of contact type* on the conical end, i.e. $J_t$ becomes time-independent and satisfies $dr \circ J_t = -\theta_M$ for $r \gg 1$. The usual Floer differential $d : SC^*(M) \to SC^{*+1}(M)$ is defined by counting rigid $J_t$-holomorphic cylinders $u : Z \to M$ with asymptotics at Hamiltonian orbits $y^+, y^- \in \mathcal{O}_M$, which can be regarded as the special case $k = 0$ of the operations $\delta_k$ on the complex $SC^*(M)$ defined below.

**Definition 7.1.2** (Definition 13 of [43]). *A Floer datum for a k-point angle-decorated cylinder $(Z, p_1, \cdots, p_k)$ consists of the following:*

- *the choices of positive and negative cylindrical ends for $Z$, as in (7.4);*
- *a 1-form $\alpha_Z = dt$ on $Z$;*
- *a surface-dependent Hamiltonian function $H_Z : Z \to \mathcal{H}(M)$ which satisfies*

$$(\varepsilon^{\pm})^* H_Z = H_t, \tag{7.13}$$

  *where $H_t \in \mathcal{H}(M)$ is some fixed choice of an admissible Hamiltonian;*

- *a surface-dependent almost complex structure $J_Z : Z \to \mathcal{J}(M)$ such that*

$$(\varepsilon^{\pm})^* J_Z = J_t \tag{7.14}$$

  *for some fixed choice of $J_t \in \mathcal{J}(M)$.*

*Universal and consistent* choices of Floer data over the compactified moduli spaces $\overline{\mathcal{M}}_k$ for all $k \geqslant 1$ can be constructed in an inductive way. In our specific situation, this means that (cf. Definition 14 of [43])

- At the boundary strata (7.7), the choices of Floer data should coincide with the product of the Floer data chosen on lower dimensional moduli spaces. Moreover, the choices vary smoothly with respect to the gluing charts.
- At a boundary stratum of the form (7.8), the Floer datum chosen for a representative $(Z, p_1, \cdots, p_k)$ of an element of $\overline{\mathcal{M}}_k^{i,i+1}$ coincides with the one pulled back from the corresponding element of $\overline{\mathcal{M}}_{k-1}$ via the forgetful map $\bar{\pi}_i$ up to conformal equivalence.

Here we say that two Floer data

$$F_1 = (\varepsilon_1^{\pm}, \alpha_{Z,1}, H_{Z,1}, J_{Z,1}) \text{ and } F_2 = (\varepsilon_2^{\pm}, \alpha_{Z,2}, H_{Z,2}, J_{Z,2}) \tag{7.15}$$

for $(Z, p_1, \cdots, p_k)$ are *conformally equivalent* if the choices of strip like ends $\varepsilon_1^{\pm}$ and $\varepsilon_2^{\pm}$ coincide, and for some constant $c > 0$,

$$H_{t,1} = \frac{H_{t,2}}{c} \circ \psi^c \tag{7.16}$$

and

$$J_{t,1} = (\psi^c)^* J_{t,2} \tag{7.17}$$

hold on the cylindrical ends, with $\psi^c$ being the time-$c$ Liouville flow, see Definition 9 of [43].

Fix a universal and consistent choice of Floer data for $(Z, p_1, \cdots, p_k) \in \mathcal{M}_k$. For any pair of orbits $y^+, y^- \in \mathcal{O}_M$ and any integer $k \geqslant 1$, we introduce the moduli space $\mathcal{M}_k(y^+; y^-)$ of pairs

$$((Z, p_1, \cdots, p_k), u), \tag{7.18}$$

where $(Z, p_1, \cdots, p_k) \in \mathcal{M}_k$ and $u : Z \to M$ is a map which satisfies Floer's equation

$$(du - X_{H_Z} \otimes dt)^{0,1} = 0, \tag{7.19}$$

where the $(0,1)$-part is taken with respect to the domain-dependent almost complex structure $J_Z$ chosen above as part of our Floer data, together with the asymptotic conditions

$$\lim_{s \to \pm\infty} (\varepsilon^{\pm})^* u(s, \cdot) = y^{\pm}. \tag{7.20}$$

The boundary of the Gromov compactification $\overline{\mathcal{M}}_k(y^+; y^-)$ is covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_j(y; y^-) \times \overline{\mathcal{M}}_{k-j}(y^+; y) \hookrightarrow \partial \overline{\mathcal{M}}_k(y^+; y^-), \tag{7.21}$$

$$\overline{\mathcal{M}}_k^{i,i+1}(y^+;y^-) \hookrightarrow \partial\overline{\mathcal{M}}_k(y^+;y^-), \tag{7.22}$$

which come from the boundary strata of $\overline{\mathcal{M}}_k$, along with the boundary components coming from the usual semi-stable strip breaking

$$\overline{\mathcal{M}}_k(y;y^-) \times \overline{\mathcal{M}}(y^+;y) \hookrightarrow \partial\overline{\mathcal{M}}_k(y^+;y^-), \tag{7.23}$$

$$\overline{\mathcal{M}}(y;y^-) \times \overline{\mathcal{M}}_k(y^+;y) \hookrightarrow \partial\overline{\mathcal{M}}_k(y^+;y^-). \tag{7.24}$$

For generic choices of Floer data, the moduli spaces $\overline{\mathcal{M}}_k(y^+;y^-)$ are compact manifolds-with-corners of dimension

$$\deg(y^+) - \deg(y^-) + 2k - 1. \tag{7.25}$$

Let $((Z, p_1, \cdots, p_k), u)$ be a rigid element of $\overline{\mathcal{M}}_k(y^+;y^-)$, so we have $\deg(y^+) = \deg(y^-) - 2k + 1$. In this case, there are natural isomorphisms

$$\mu_u : o_{y^-} \longrightarrow o_{y^+} \tag{7.26}$$

between the orientation lines defined via index theory. One defines the operation

$$\delta_k : SC^*(M) \to SC^{*-2k+1}(M) \tag{7.27}$$

by a signed count of rigid elements of the moduli spaces $\overline{\mathcal{M}}_k(y^+;y^-)$ for varying asymptotics $y^+$ and $y^-$. Our choices of Floer data ensures that the elements of $\overline{\mathcal{M}}_k^{i,i+1}(y^+,y^-)$ are never rigid, from which the identity

$$\sum_{i=0}^{k} \delta_i \delta_{k-i} = 0 \tag{7.28}$$

follows, see Lemma 10 of [43] for details. This shows that $(SC^*(M), \{\delta_k\}_{k\geqslant 0})$ is an $S^1$-complex (usually not strict) in the sense of Definition 2.6.1.

As in (2.56), the $S^1$-*equivariant symplectic cohomology* $SH^*_{S^1}(M)$ is defined to be the cohomology of the complex

$$(SC^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]], \delta_{eq})\,, \tag{7.29}$$

where $\delta_{eq} := \sum_{k=0}^{\infty} \delta_k u^k$ and $u$ is a formal variable of degree 2.

One can replace the autonomous Hamiltonian $H$ in the above construction (which is quadratic on the conical end) with a Hamiltonian which is linear at infinity, i.e.

$$H_\lambda(r, y) = \lambda r + C \tag{7.30}$$

on $[r_0, \infty) \times \partial\overline{M}$, where $\lambda \in \mathbb{R}$ and $C$ is some constant, and consider its time-dependent perturbation $H_{\lambda,t} = H_\lambda + F_{\lambda,t}$ as above, so that all the time-1 orbits of $X_{H_{\lambda,t}}$ are non-degenerate. The space of such Hamiltonians $H_{\lambda,t} : S^1 \times M \to \mathbb{R}$ with varying $\lambda$ will be denoted by $\mathcal{H}_\ell(M)$, which contains $\mathcal{H}_\lambda(M)$, the space of Hamiltonians with fixed slope $\lambda$ at infinity, as a subset, i.e.

$$\mathcal{H}_\ell(M) = \bigcup_{\lambda \in \mathbb{R}} \mathcal{H}_\lambda(M). \tag{7.31}$$

We also require that $\lambda \notin \mathcal{P}_M$, where $\mathcal{P}_M \subset \mathbb{R}$ is the collection of those $\lambda$ such that the set of 1-periodic orbits $\mathcal{O}_{M,\lambda}$ of $X_{H_{\lambda,t}}$ is not contained in any compact subset of $M$. This gives rise to the $S^1$-equivariant Floer cochain complex

$$CF^*_{S^1}(\lambda) := (CF^*(\lambda) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]], \delta_{eq})\,, \tag{7.32}$$

where $CF^*(\lambda)$ is the Hamiltonian Floer complex of $H_{\lambda,t}$. The cohomology of (7.32) will be denoted by $HF^*_{S^1}(\lambda)$. For $\lambda_1 < \lambda_2$, one can build equivariant continuation maps (by counting $k$-point angle decorated cylinders with different Floer data on its ends)

$$\kappa_{eq}^{\lambda_1,\lambda_2} := \sum_{k=0}^{\infty} \kappa_k^{\lambda_1,\lambda_2} u^k : CF^*_{S^1}(\lambda_1) \to CF^*_{S^1}(\lambda_2), \tag{7.33}$$

which are $S^1$-complex homomorphisms relating different equivariant Floer complexes. Passing to the direct limit yields an alternative definition of the equivariant symplectic cohomology

$$SH^*_{S^1}(M) := \varinjlim_\lambda HF^*_{S^1}(\lambda). \tag{7.34}$$

This approach of defining $SH^*_{S^1}(M)$ is carried out in detail in [19]. Note that for the purpose of defining $SH^*_{S^1}(M)$, one can assume that $\lambda > 0$.

Back to our previous set up. Choosing $H_t \in \mathcal{H}(M)$ to be a $C^2$-small and time-independent Morse function in the interior $M^{in}$ of $\overline{M}$, we get a copy of the Morse complex $CM^*(H_t)$ as a subcomplex of $SC^*(M)$. When passing to cohomology, the inclusion $CM^*(H_t) \hookrightarrow SC^*(M)$ induces the classical PSS map

$$PSS : H^*(M;\mathbb{K}) \to SH^*(M). \tag{7.35}$$

Since the $S^1$-action on the symplectic cochain complex $SC^*(M)$ comes from reparametrizing the Hamiltonian orbits, one may expect that it becomes trivial when restricted to the subcomplex $CM^*(H_t)$, i.e. all the operations $\delta_k, k \geqslant 1$ should vanish on it. This is indeed the case with appropriate choices of Floer data over the moduli spaces $\overline{\mathcal{M}}_k$. As a consequence, the natural inclusion $CM^*(H_t) \hookrightarrow SC^*(M)$ is an $S^1$-complex homomorphism, which induces the $S^1$-*equivariant PSS map*

$$\widetilde{PSS} : H^*(M;\mathbb{K}((u))/u\mathbb{K}[[u]]) \to SH^*_{S^1}(M) \tag{7.36}$$

after passing to cohomologies.

One can also use a $H_{\lambda,t} \in \mathcal{H}_\lambda(M)$ which is $C^2$-small and Morse in $M^{in}$ in the above, which will then give rise to maps

$$PSS_\lambda : H^*(M;\mathbb{K}) \to HF^*(\lambda), \tag{7.37}$$

$$\widetilde{PSS}_\lambda : H^*(M;\mathbb{K}((u))/u\mathbb{K}[[u]]) \to HF^*_{S^1}(\lambda). \tag{7.38}$$

The maps (7.35) and (7.36) can then be thought of as (7.37) and (7.38) composed with the continuation maps and equivariant continuation maps respectively.

## 7.2 Cyclic dilations

We study in this section the geometric counterpart of an exact Calabi-Yau structure on a homologically smooth $A_\infty$-category. The precise relationship between these two notions will be established in the next section, using Ganatra's construction of the cyclic open-closed string map.

Take any $\lambda \in [0,\infty]$ with $\lambda \notin \mathcal{P}_M$ if $\lambda < \infty$. Recall that the $S^1$-equivariant Hamiltonian Floer cohomology $HF^*_{S^1}(\lambda)$ fits into the following Gysin type long exact sequence:

$$\cdots \to HF^{*-1}(\lambda) \xrightarrow{\mathbf{I}} HF^{*-1}_{S^1}(\lambda) \xrightarrow{\mathbf{S}} HF^{*+1}_{S^1}(\lambda) \xrightarrow{\mathbf{B}} HF^*(\lambda) \to \cdots, \tag{7.39}$$

see [18, 19, 20] for a detailed discussion. The BV operator

$$\Delta = [\delta_1] : HF^*(\lambda) \to HF^{*-1}(\lambda) \tag{7.40}$$

coincides on the cohomology level with the composition $\mathbf{B} \circ \mathbf{I}$ (in the non-trivial order). When $M = T^*Q$ is the cotangent bundle of a compact *Spin* manifold $Q$, and $\lambda = \infty$, the above Gysin exact sequence reduces to the well-known long exact sequence in string topology [26] (this is because the Cieliebak-Latschev map is a morphism between $S^1$-complexes, see Section 8.1):

$$\cdots \to H_{*+1}(\mathcal{L}Q;\mathbb{K}) \xrightarrow{\mathbf{I}} H^{S^1}_{*+1}(\mathcal{L}Q;\mathbb{K}) \xrightarrow{\mathbf{S}} H^{S^1}_{*-1}(\mathcal{L}Q;\mathbb{K}) \xrightarrow{\mathbf{B}} H_*(\mathcal{L}Q;\mathbb{K}) \to \cdots, \tag{7.41}$$

where the maps $\mathbf{I}$ and $\mathbf{B}$ are known as the *erasing map* and the *marking map* respectively. Because of this, the same terminology will be used to refer to $\mathbf{I}$ and $\mathbf{B}$ in the general case, for any $\lambda$ and any Liouville manifold $M$.

Recall that in string topology, the map $\mathbf{B}$ is defined as follows: by definition, an equivariant homology class $\gamma \in H^{S^1}_{-*+1}(\mathcal{L}Q;\mathbb{K})$ can be equivalently realized as a homology class of $H_{*-1}(S^\infty \times_{S^1} \mathcal{L}Q;\mathbb{K})$, whose lift in $H_*(S^\infty \times \mathcal{L}Q;\mathbb{K})$ will be denoted by $\tilde{\gamma}$. Let $\Pi$ be the trivial projection $S^\infty \times \mathcal{L}Q \to \mathcal{L}Q$, then $\mathbf{B}(\gamma) = \Pi_*(\tilde{\gamma})$.

Since the natural inclusion $CM^*(H_t) \hookrightarrow SC^*(M)$ is an $S^1$-complex homomorphism, by Proposition 2.6.2 the long exact sequence (7.39) is compatible with the ordinary and equivariant PSS maps (cf. (7.37) and (7.38)) in the sense that it fits into the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{*-1}(M;\mathbb{K}) \arrow[r] \arrow[d, "PSS_\lambda"] & H^{*-1}_{S^1}(M;\mathbb{K}) \arrow[r] \arrow[d, "\widetilde{PSS}_\lambda"] & H^{*+1}_{S^1}(M;\mathbb{K}) \arrow[r] \arrow[d, "\widetilde{PSS}_\lambda"] & H^*(M;\mathbb{K}) \arrow[r] \arrow[d, "PSS_\lambda"] & \cdots \\
\cdots \arrow[r] & HF^{*-1}(\lambda) \arrow[r, "\mathbf{I}"] & HF^{*-1}_{S^1}(\lambda) \arrow[r, "\mathbf{S}"] & HF^{*+1}_{S^1}(\lambda) \arrow[r, "\mathbf{B}"] & HF^*(\lambda) \arrow[r] & \cdots
\end{tikzcd} \tag{7.42}$$

where the upper row is the usual Gysin long exact sequence for the trivial $S^1$-action on $M$, so in particular $H^*_{S^1}(M;\mathbb{K}) \cong H^*(M;\mathbb{K}((u))/u\mathbb{K}[[u]])$.

We proceed to give a chain level interpretation of the condition (1.41) appeared in Section 1.3. To do this, we need to find the chain level expression of the coboundary map

$$\mathbf{B} : HF^1_{S^1}(\lambda) \to HF^0(\lambda) \tag{7.43}$$

in the Gysin sequence (7.39). Write a degree 1 cocycle $\tilde{\beta}$ in the $S^1$-equivariant Hamiltonian Floer cochain complex $CF^*(\lambda) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]]$ as $\sum_{k=0}^\infty \beta_k \otimes u^{-k}$, where $\beta_k \in CF^{2k+1}(\lambda)$ and only finitely many terms in the infinite sum are non-zero. If we use $\mathbf{B}_c$ to denote the underlying chain level map of the marking map $\mathbf{B}$, standard diagram chasing argument enables us to find that (see Proposition 2.9 of [19])

$$\mathbf{B}_c\left(\sum_{k=0}^{\infty}\beta_k\otimes u^{-k}\right)=\sum_{k=0}^{\infty}\delta_{k+1}(\beta_k). \tag{7.44}$$

We introduce the following definition, which is slightly more general than (1.41), and will be useful for various constructions and arguments later.

**Definition 7.2.1.** *A cyclic dilation is a cocycle $\tilde{\beta}\in SC^1_{S^1}(M)$ which consists of a sequence of odd degree Floer cochains $\{\beta_k\}_{k\geqslant 0}$ with $\beta_k\in SC^{2k+1}(M)$, and $\beta_k\neq 0$ for only finitely many $k$, so that the cocycle $\sum_{k=0}^{\infty}\delta_{k+1}(\beta_k)$, after passing to cohomology, defines an invertible element $h\in SH^0(M)^\times$. Moreover, there exists a $\lambda\in\mathbb{R}_{>0}\backslash\mathcal{P}_M$ so that $\tilde{\beta}$ lies in the image of the equivariant continuation map (cf. (7.33))*

$$\kappa_{eq}^{\lambda,\infty}:CF^1_{S^1}(\lambda)\to CF^1_{S^1}(\infty):=SC^1_{S^1}(M). \tag{7.45}$$

Let $\lambda\gg 0$ be sufficiently large. We consider an important special case of the above definition, namely when $h=1$ is the identity of $SH^0(M)$. It follows from the exactness of (7.39) that there is a cohomology class $\tilde{b}\in HF^1_{S^1}(\lambda)$ satisfying $\mathbf{B}(\tilde{b})=1$ if and only if $1\in HF^0(\lambda)$ vanishes under the erasing map $\mathbf{I}:HF^0(\lambda)\to HF^0_{S^1}(\lambda)$.

In view of the commutative diagram (7.42), this is precisely the case when the image of the (locally finite) fundamental class $1\in H^0(M;\mathbb{K})$ vanishes under the composition

$$H^*(M;\mathbb{K})\hookrightarrow H^*(M;\mathbb{K}((u))/u\mathbb{K}[[u]])\xrightarrow{\widetilde{PSS}_\lambda}HF^*_{S^1}(\lambda). \tag{7.46}$$

On the other hand, it follows from Lemma 4.2.4 of [117] that if $M$ admits a dilation in $HF^1(\lambda)$, i.e. a class $b\in HF^1(\lambda)$ which becomes a dilation in $SH^1(M)$ under the continuation map $\kappa^{\lambda,\infty}$ considered above, then $1\in H^0(M;\mathbb{K})$ lies in the kernel of

(7.46). This observation enables us to relate cyclic dilations to the following notion introduced by Zhao [117].

**Definition 7.2.2** (Definition 4.2.1 of [117])**.** *We say that a Liouville manifold $M$ admits a higher dilation if the identity $1 \in H^*(M;\mathbb{K})$ lies in the kernel of the localized equivariant PSS map*

$$\widehat{PSS} : H^*(M;\mathbb{K}((u))) \to \widehat{PSH}^*(M), \tag{7.47}$$

*where $\widehat{PSH}^*(M)$ is the completed periodic symplectic cohomology, which is the cohomology of the Tate complex (cf. (2.57))*

$$(SC^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u)), \delta_{eq}) \, . \tag{7.48}$$

A higher dilation can be equivalently interpreted using the equivariant PSS map (7.36). Precisely, $M$ admits a higher dilation if and only if $1 \otimes u^{-k} \in H^*(M;\mathbb{K}((u))/u\mathbb{K}[[u]])$ lies in the kernel of the map $\widetilde{PSS}$ (cf. (7.36)) for every $k \geqslant 0$. In view of the above discussions, we get the following:

**Proposition 7.2.1.** *If $M$ admits a higher dilation, then it admits a cyclic dilation with $h = 1$.*

**Remark 7.2.1.** *In fact, it is an observation made in Remark 6.5 of* [*103*] *that the existence of a dilation in $HF^1(\lambda)$ is equivalent to the existence of a cocycle $\beta_0 \in CF^1(\lambda)$ and a cochain $\beta_{-1} \in CF^{-1}(\lambda)$, so that $\delta_{eq}(\beta_{-1} + \beta_0 \otimes u^{-1}) = e$, where $e \in CF^0(\lambda)$ is the chain level representative of the identity. It is therefore natural to consider a sequence of cochains $\{\beta_j\}_{j \geqslant -1}$ with arbitrary length, where $\beta_j \in CF^{2j+1}(\lambda)$, so that*

$$\delta_{eq}\left(\sum_{j=-1}^{\infty} \beta_j \otimes u^{-j-1}\right) = e, \tag{7.49}$$

*where only finitely many $\beta_j$ can be nonzero. While the notion of a higher dilation allows this more general situation to happen, it also imposes the additional restriction that $e \otimes u^{-k}$ should be coboundaries in the complexes $CF_{S^1}^{-2k}(\lambda)$ for $\lambda \gg 0$. (7.49) holds if and only if $M$ admits a cyclic dilation with $h = 1$.*

This observation enables us to get some first examples of Liouville manifolds with cyclic dilations. For any closed manifold $Q$, one can consider the classifying map $f : Q \to B\pi_1(Q)$ for its universal cover. $Q$ is called *rationally inessential* if the fundamental class $[Q] \in H_n(Q;\mathbb{Q})$ vanishes under the pushforward

$$f_* : H_n(Q;\mathbb{Q}) \to H_n(B\pi_1(Q);\mathbb{Q}). \tag{7.50}$$

In particular, every simply connected closed manifold is rationally inessential. It follows from Corollary 1.1.6 of [117] and Proposition 7.2.1 stated above that for any rationally inessential manifold $Q$, $M = T^*Q$ admits a cyclic dilation over $\mathbb{Q}$. It is, however, not clear whether such a cotangent bundle admits a quasi-dilation. In fact, it is even unknown whether $T^*Q$ admits a dilation over $\mathbb{Q}$ for any simply connected *formal* manifold $Q$, see Lecture 18 of [93]. More interesting examples of Liouville manifolds which admit cyclic dilations are established in Section 9.1.

**Remark 7.2.2.** *Related notions are introduced in [119], where the author considers the spectral sequence associated to the u-adic filtration on the equivariant Floer cochain complex $CF_{S^1}^*(\lambda)$, and $M$ is said to admit a $k$-dilation if for sufficiently large $\lambda \notin \mathcal{P}_M$, the identity $e \in CF^0(\lambda)$ is killed in the $(k+1)$-th page of the spectral sequence. In particular, any flexible Weinstein manifold admits a 0-dilation, and a dilation in the sense of Seidel-Solomon [103] is a 1-dilation. In general, having a $k$-dilation for $k \geqslant 1$ is equivalent to requiring that $\sum_{j=0}^{k-1} \beta_j \otimes u^{-j} \in CF_{S^1}^1(\lambda)$ defines a cyclic dilation with $h = 1$.*

## 7.3 Cyclic open-closed map

We briefly summarize the construction of the cyclic open-closed string map due to Ganatra [43]. Details can be found in Section 5 of [43]. See also [46] for applications of the cyclic open-closed string map in the study of mirror symmetry of closed symplectic manifolds.

Roughly speaking, the cyclic open-closed map is a parametrized version of the usual open-closed map

$$OC : CH_*(\mathcal{W}(M)) \to SC^{*+n}(M) \tag{7.51}$$

considered in [94] and [2], which keeps track of the $S^1$-complex structures on both sides.

However, as we have already noticed in Section 2.7, in order to keep track of the information of the $S^1$-action on the open-string side, one needs to consider the non-unital Hochschild complex $CH_*^{nu}(\mathcal{W}(M))$ instead of the usual Hochschild complex $CH_*(\mathcal{W}(M))$. Thus the first step towards the construction of an "$S^1$-equivariant enhancement" of the usual open-closed map $OC$ would be to replace $OC$ by a map

$$OC^{nu} : CH_*^{nu}(\mathcal{W}(M)) \to SC^{*+n}(M) \tag{7.52}$$

defined on the non-unital Hochschild complex. Following Ganatra, we will call $OC^{nu}$ the *non-unital open-closed string map*. In view of the definition of $CH_*^{nu}(\mathcal{W}(M))$ recalled in Section 2.7, the map $OC^{nu}$ should consist of the check component $\widecheck{OC} : CH_*(\mathcal{W}(M)) \to SC^{*+n}(M)$ and the hat component $\widehat{OC} : CH_*(\mathcal{W}(M))[1] \to SC^{*+n}(M)$, which acts respectively on the check and hat factors of the non-unital Hochschild complex, and

$$OC^{nu}(\check{\alpha}, \hat{\beta}) = \widecheck{OC}(\check{\alpha}) + \widehat{OC}(\hat{\beta}), \tag{7.53}$$

where $\check{\alpha} \in CH_*(\mathcal{W}(M))$ and $\hat{\beta} \in CH_*(\mathcal{W}(M))[1]$.

The map $\widetilde{OC}$ is defined on the ordinary Hochschild chain complex, and it in fact coincides with the ordinary open-closed string map (7.51). Recall that $OC$ is defined by considering closed discs $\overline{S}$ equipped with boundary marked points $\zeta_1, \cdots, \zeta_d \in \partial\overline{S}$ which serve as inputs, and an interior marked point $\zeta_{out}$, which is an output. There is also an asymptotic marker $\ell_{out}$ at $\zeta_{out}$ pointing towards $\zeta_d$. One can assign Floer data to such discs $\left(\overline{S}; \zeta_1, \cdots, \zeta_d; \zeta_{out}, \ell_{out}\right)$ in the usual way, and when forming the moduli space of Floer trajectories, the boundary components of $\partial\overline{S}\backslash\{\zeta_1, \cdots, \zeta_d\}$ will be labelled with Lagrangian submanifolds $L_1, \cdots, L_d$ which are objects of the wrapped Fukaya category $\mathcal{W}(M)$, so that $L_i$ is the label of the arc along the boundary between $\zeta_i$ and $\zeta_{i+1 \bmod d}$, and the marked points $\zeta_1, \cdots, \zeta_d$ are associated with asymptotics $x_1, \cdots, x_d$, which are time-1 chords of the Hamiltonian vector field $X_{H_t}$ from $L_{i-1}$ to $L_{i \bmod d}$, for some $H_t \in \mathcal{H}(M)$. For any $y_{out} \in \mathcal{O}_M$, the coefficient before $|o_{y_{out}}|_{\mathbb{K}}$ in $\widetilde{OC}\left(|o_{x_d}|_{\mathbb{K}}, \cdots, |o_{x_1}|_{\mathbb{K}}\right)$ is determined by a signed count of rigid Floer trajectories $u : S \to M$ which satisfy the relevant Floer equation, with boundary conditions determined by the Lagrangian labellings $(L_1, \cdots, L_d)$ and asymptotic conditions specified by $(\vec{x} := (x_d, \cdots, x_1); y_{out})$.

The definition of the map $\widehat{OC}$ differs from $\widetilde{OC}$ in the sense that one now considers closed discs $\left(\overline{S}; \zeta_f, \zeta_1, \cdots, \zeta_d; \zeta_{out}, \ell_{out}\right)$ with $d+1$ boundary marked points, and an interior marked point as the domains, where $\zeta_f$ is an auxiliary marked point on the boundary, and the asymptotic marker $\ell_{out}$ is required to point towards $\zeta_f$. The marked point $\zeta_f$ is auxiliary in the sense that this point will be forgotten when assigning Floer data, and the collection $(\vec{x}; y_{out})$ of Hamiltonian chords and orbits as above still determines the asymptotic conditions for the corresponding Floer equation. Since the direction of $\ell_{out}$ remembers the position of $\zeta_f$, so its freedom to vary increases the degree of the map by 1, which explains why $\widehat{OC}$ is a map defined on the

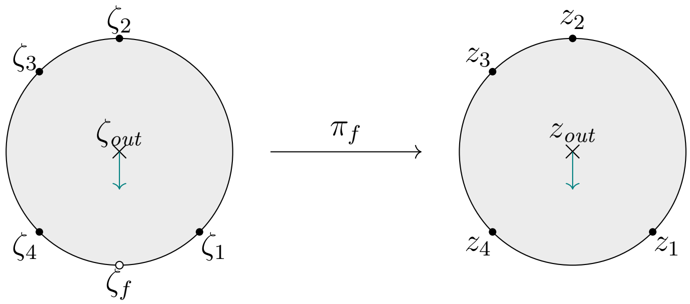


Figure 7.1: Domain of the map $\widehat{OC}$ and its identification after forgetting $\zeta_f$

shifted Hochschild chain complex $CH_*(\mathcal{W}(M))[1]$. See Figure 7.1 for a description of its domain.

**Proposition 7.3.1** (Lemma 12 of [43]). *The non-unital open-closed map $OC^{nu} = \widetilde{OC} \oplus \widehat{OC}$ is a chain map.*

Consider the natural inclusion $\iota : CH_*(\mathcal{W}(M)) \hookrightarrow CH_*^{nu}(\mathcal{W}(M))$, whose composition with the non-unital open-closed map gives rise to a chain map $OC^{nu} \circ \iota : CH_*(\mathcal{W}(M)) \to SC^{*+n}(M)$, which coincides on the chain level with the usual open-closed string map $OC$. Since we have learned from Section 2.7 that $\iota$ is a quasi-isomorphism, it follows that as homology level maps, $[OC^{nu}] \circ \iota = [OC]$.

The cyclic open-closed string map $\widetilde{OC}$ will be defined as an $S^1$-equivariant enhancement of $OC^{nu}$, by including higher cyclic chain homotopies. More precisely, it consists of a sequence of maps

$$OC^k = \widetilde{OC}^k \oplus \widehat{OC}^k : CH_*^{nu}(\mathcal{W}(M)) \to SC^{*+n-2k}(M) \tag{7.54}$$

for each $k \geqslant 0$, such that $\widetilde{OC}^0 = \widetilde{OC}$, $\widehat{OC}^0 = \widehat{OC}$, and for any $k \geqslant 1$, we have

$$(-1)^n \sum_{i=0}^{k} \delta_i \circ \widetilde{OC}^{k-i} = \widehat{OC}^{k-1} \circ \mathbb{B}^{nu} + \widetilde{OC}^k \circ b, \tag{7.55}$$

$$(-1)^n \sum_{i=0}^{k} \delta_i \circ \widehat{OC}^{k-i} = \widehat{OC}^k \circ b' + \widetilde{OC}^k \circ (1-\lambda), \tag{7.56}$$

where

$$\mathbb{B}^{nu} : CH_*^{nu}(\mathcal{W}(M)) \to CH_{*-1}^{nu}(\mathcal{W}(M)) \tag{7.57}$$

is the map (2.66) applied to the wrapped Fukaya category. Roughly speaking, the maps $\widetilde{OC}^k$ and $\widehat{OC}^k$ are defined in the same way as $\widetilde{OC}$ and $\widehat{OC}$, but with additional interior marked points $p_1, \cdots, p_k$ included in the respective domains, which are located near $\zeta_{out}$ and are strictly radially ordered in the sense that

$$0 < |p_1| < \cdots < |p_k| < \frac{1}{2}. \tag{7.58}$$

Define

$$\widetilde{OC} := \sum_{k=0}^{\infty} \left( \widetilde{OC}^k \oplus \widehat{OC}^k \right) u^k, \tag{7.59}$$

it follows from (7.55) and (7.56) that:

**Theorem 7.3.1** (Theorem 1 of [43]). *The non-unital open-closed map $OC^{nu}$ admits a geometrically defined $S^1$-equivariant enhancement*

$$\widetilde{OC} \in R\mathrm{Hom}_{S^1}\left( CH_*^{nu}(\mathcal{W}(M))[n], SC^*(M) \right). \tag{7.60}$$

Combining Theorem 1.1 of [44] and Corollary 1 of [43], we have the following:

**Theorem 7.3.2** (Ganatra). *Let $M$ be a non-degenerate Liouville manifold, then the homology level maps*

$$[OC] : HH_*(\mathcal{W}(M)) \to SH^{*+n}(M), [\widetilde{OC}] : HC_*(\mathcal{W}(M)) \to SH_{S^1}^{*+n}(M) \tag{7.61}$$

*are isomorphisms.*

We can now fulfil our promise at the beginning of Section 7.2, namely to explain the relationship between exact Calabi-Yau structures on $\mathcal{W}(M)$ and cyclic dilations.

**Proposition 7.3.2.** *For any Liouville manifold $M$, there is a commutative diagram*

$$\begin{tikzcd} HC_{*+1}(\mathcal{W}(M)) \arrow[r, "\mathbb{B}"] \arrow[d, "{[\widetilde{OC}]}"'] & HH_*(\mathcal{W}(M)) \arrow[d, "{[OC]}"] \\ SH^{*+n+1}_{S^1}(M) \arrow[r, "\mathbf{B}"] & SH^{*+n}(M) \end{tikzcd} \tag{7.62}$$

*where $\mathbb{B}$ is the cohomology level map associated to $\mathbb{B}^{nu}$.*

*Proof.* This is a direct consequence of Theorem 7.3.1 and Proposition 2.6.2. □

**Corollary 7.3.1.** *Let $M$ be a non-degenerate Liouville manifold, its wrapped Fukaya category $\mathcal{W}(M)$ is exact Calabi-Yau if and only if there exists a cyclic dilation $\tilde{b} \in SH^1_{S^1}(M)$.*

*Proof.* Since $M$ is non-degenerate, it follows from Theorem 7.3.2 that both of the maps $[OC]$ and $[\widetilde{OC}]$ in the commutative diagram (7.62) are isomorphisms. The corollary then follows from the fact that $[\eta] \in HH_{-n}(\mathcal{W}(M))$ is non-degenerate if and only if its image under the open-closed map $[OC]$ is an invertible element $h \in SH^0(M)^\times$, which is a corollary of the proof Theorem 3 of [43]. □

# Chapter 8

# Lagrangian submanifolds

Let $M$ be a Liouville manifold with $c_1(M) = 0$, and fix a trivialization of its canonical bundle $K_M$. We consider in this chapter the open string implications of the existence of a cyclic dilation. To be precise, we shall consider Lagrangian submanifolds in $M$ which are objects of the compact Fukaya category $\mathcal{F}(M)$, namely they satisfy the following:

**Assumption 8.0.1.** *$L \subset M$ is closed, connected, exact, graded, and Spin.*

We shall actually fix the choice of a grading on $L$, so that the Lagrangian Floer cohomology $HF^*(L_0, L_1)$ of two Lagrangian submanifolds $L_0, L_1 \subset M$ is well-defined as a $\mathbb{Z}$-graded algebra over $\mathbb{K}$.

## 8.1 The Cieliebak-Latschev map

Let $L \subset M$ be an exact Lagrangian submanifold satisfying Assumption 8.0.1. As a consequence of the Viterbo functoriality [113, 3], we have a map

$$SH^*(M) \to SH^*(T^*L) \cong H_{-*}(\mathcal{L}L; \nu), \tag{8.1}$$

where the latter isomorphism is established in [1] in the case when $L$ is *Spin*, and in [3] in the general case. Since we have required in Assumption 8.0.1 that $L$ is *Spin*, the local system $\nu : \pi_1(\mathcal{L}L) \to \mathbb{K}$ can be dropped out from our notations.

There is an $S^1$-equivariant analogue of (8.1) constructed by Cohen-Ganatra [29] (in the case when $M = T^*L$, see also Section 4.4.1 of [117] for a detailed exposition in the general case), which is an infinite sum

$$\widetilde{CL} := \sum_{k=0}^{\infty} CL_k u^k : SC^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]] \to C^{\diamondsuit}_{n-*}(\mathcal{L}L; \mathbb{K}((u))/u\mathbb{K}[[u]]) \quad (8.2)$$

whose degree 0 piece arises from relevant considerations by Cieliebak-Latschev in [25]. In the above, $C^{\diamondsuit}_{-*}(\mathcal{L}L; \mathbb{K})$ is a quotient of the dg algebra $C_{-*}(\mathcal{L}L; \mathbb{K})$ constructed by Cohen-Ganatra in Appendix A.1 of [29]. It has the property that the projection $C_{-*}(\mathcal{L}L; \mathbb{K}) \to C^{\diamondsuit}_{-*}(\mathcal{L}L; \mathbb{K})$ is a quasi-isomorphism, and $C^{\diamondsuit}_{-*}(\mathcal{L}L; \mathbb{K})$ carries the structure of a strict $S^1$-complex. $\widetilde{CL}$ defines an $S^1$-complex morphism, so it descends to the map (1.42) on the cohomology level. We shall give a brief account of Cohen-Ganatra's construction in this section, and explain its implications for Lagrangian submanifolds in Liouville manifolds with cyclic dilations.

The construction of the maps $\{CL_k\}$ is in some sense parallel to the construction of the maps $\{\delta_k\}$ in Section 7.1, but we now consider half-cylinders instead of cylinders as our domains. A *k-point angle decorated half-cylinder* is a (positive) half-cylinder $Z^+ \subset Z$ together with a collection of auxiliary interior marked points $p_1, \cdots, p_k \in Z^+$ satisfying (7.1). Denote by $\mathcal{M}_{k,+}$ the moduli space of such half-cylinders. Every element of $\mathcal{M}_{k,+}$ is equipped with a positive cylindrical end

$$\varepsilon^+ : [0, \infty) \times S^1 \to Z^+, (s,t) \mapsto (s + (p_k)_s + \delta, t), \quad (8.3)$$

for some fixed $\delta > 0$. Note that unlike the case of $\mathcal{M}_k$, there is no free $\mathbb{R}$-action on the moduli space $\mathcal{M}_{k,+}$.

$\mathcal{M}_{k,+}$ can be compactified to a manifold with corners $\overline{\mathcal{M}}_{k,+}$ by including broken trajectories in the moduli space. The codimension 1 boundary strata of $\overline{\mathcal{M}}_{k,+}$ can be covered by the images of the natural inclusions

$$\overline{\mathcal{M}}_{j,+} \times \overline{\mathcal{M}}_{k-j} \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}, 0 \leqslant j \leqslant k, \tag{8.4}$$

$$\overline{\mathcal{M}}_{k,+}^{i,i+1} \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}, 1 \leqslant i < k, \tag{8.5}$$

$$\overline{\mathcal{M}}_{k,+}^{0} \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}, \tag{8.6}$$

where $\mathcal{M}_{k,+}^{i,i+1}$ is the locus where the $i$-th and $(i+1)$-th height coordinates coincide, and $\mathcal{M}_{k,+}^{0}$ is the locus where $h_1 = 0$. There exist forgetful maps

$$\pi_i : \mathcal{M}_{k,+}^{i,i+1} \to \mathcal{M}_{k-1,+}, 1 \leqslant i \leqslant k-1, \tag{8.7}$$

$$\pi_0 : \mathcal{M}_{k,+}^{0} \to \mathcal{M}_{k-1,+}, \tag{8.8}$$

where the first map has been considered in (7.9), which forgets the point $p_{i+1}$, while the second map forgets $p_1$. Note that the maps $\pi_i$ for $i \geqslant 1$ extend as maps $\bar{\pi}_i : \overline{\mathcal{M}}_{k,+}^{i,i+1} \to \overline{\mathcal{M}}_{k-1,+}$ on the compactifications. Let $\mathcal{M}_{k,+}^{0,\theta} \subset \mathcal{M}_{k,+}^{0}$ be the subset with $\arg(p_k) = \theta$, there is an identification $\iota_\theta : \mathcal{M}_{k,+}^{0,\theta} \to \mathcal{M}_{k-1,+}$, which extends as a map $\bar{\iota}_\theta$ on the compactifications.

The definition of a Floer datum for a $k$-point angle-decorated half-cylinder $(Z^+, p_1, \cdots, p_k)$ is completely analogous to that of Definition 7.1.2, and will therefore be omitted. Inductively, there exist universal and consistent choices of Floer data for each $k \geqslant 0$ and each $k$-point angle decorated half-cylinder in the sense that:

- In a sufficiently small neighborhood of $L \subset M$, the Hamiltonian $H_{Z^+} = 0$ near $s = 0$.

- Near the boundary stratum (8.4), the Floer datum coincides with the product of the Floer data chosen on lower dimensional strata up to conformal equivalence. The Floer data vary smoothly with respect to the gluing charts for the product Floer data.

- Near the boundary strata (8.5), the Floer data are conformally equivalent to the ones obtained by pulling back from $\overline{\mathcal{M}}_{k-1,+}$ via the forgetful maps $\bar{\pi}_i$ for $i = 1, \cdots, k-1$.

Fixing a universal and consistent choice of Floer data, for each $y \in \mathcal{O}_M$, define $\mathcal{M}_{k,+}(y, L)$ to be the moduli space of pairs $((Z^+, p_1, \cdots, p_k), u)$, where $(Z^+, p_1, \cdots, p_k) \in \mathcal{M}_{k,+}$, and $u : Z^+ \to M$ is a map satisfying the parametrized Floer equation

$$\left(du - X_{H_{Z^+}} \otimes dt\right)^{0,1} = 0, \tag{8.9}$$

where the $(0,1)$-part is taken with respect to $J_{Z^+}$, together with asymptotic and boundary conditions

$$\lim_{s\to\infty} (\varepsilon^+)^* u(s, \cdot) = y, \tag{8.10}$$

$$u(0, t) = \gamma \text{ for some } \gamma \in \mathcal{L}L. \tag{8.11}$$

For generic choices of $J_{Z^+}$, $\mathcal{M}_{k,+}(y, L)$ is a smooth manifold of dimension

$$n - \deg(y) + 2k, \tag{8.12}$$

which admits a well-defined Gromov bordification $\overline{\mathcal{M}}_{k,+}(y, L)$, whose codimension 1 boundary is covered by the inclusions

$$\overline{\mathcal{M}}_{k-j,+}(y', L) \times \overline{\mathcal{M}}_j(y, y') \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}(y, L), \tag{8.13}$$

$$\overline{\mathcal{M}}^{i,i+1}_{k,+}(y, L) \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}(y, L), \tag{8.14}$$

$$\overline{\mathcal{M}}^0_{k,+}(y,L) \hookrightarrow \partial\overline{\mathcal{M}}_{k,+}(y,L). \tag{8.15}$$

Choose some Riemannian metric $g$ on $L$. The evaluation map $ev : \mathcal{M}_{k,+}(y,L) \to \mathcal{L}L$ is defined by restricting $u \in \mathcal{M}_{k,+}(y,L)$ to $\{0\} \times S^1$ and taking the arc length parametrization of the boundary of $u$ with respect to $g$. The map $ev$ admits an extension $\overline{ev} : \overline{\mathcal{M}}_{k,+}(y,L) \to \mathcal{L}L$ to the boundary strata, and the $k$-th order Cieliebak-Latschev map

$$CL_k : SC^*(M) \to C^{\diamondsuit}_{n-*+2k}(\mathcal{L}L;\mathbb{K}). \tag{8.16}$$

is defined as

$$CL_k(|o_y|_{\mathbb{K}}) = (-1)^{\deg(y)}\overline{ev}_*\left(\left[\overline{\mathcal{M}}_{k,+}(y,L)\right]\right), \tag{8.17}$$

where $\left[\overline{\mathcal{M}}_{k,+}(y,L)\right]$ denotes the fundamental chain.

**Proposition 8.1.1** ([117], Proposition 4.4.12)**.** *$\widetilde{CL} = \sum_{k=0}^{\infty} CL_k u^k$ defines a morphism of $S^1$-complexes, and therefore it is an $S^1$-equivariant enhancement of $CL_0$.*

The proof follows from an analysis of the boundary strata of $\overline{\mathcal{M}}_{k,+}(y,L)$. In particular, our choice of Floer data ensures that the elements in the moduli space $\overline{\mathcal{M}}^{i,i+1}_{k,+}(y,L)$ will never contribute. On the other hand, the contribution from the stratum $\overline{\mathcal{M}}^0_{k,+}(y,L)$ is non-trivial, and can actually be identified with

$$\delta_1^{top} \circ CL_{k-1}, \tag{8.18}$$

where $\delta_1^{top}$ denotes the chain level BV operator on $C_{-*}(\mathcal{L}L;\mathbb{K})$ defined by rotating the loops, which descends to a BV operator on the quotient dg algebra $C^{\diamondsuit}_{-*}(\mathcal{L}L;\mathbb{K})$.

On the cohomology level, $\widetilde{CL}$ induces a map

$$[\widetilde{CL}] : SH^*_{S^1}(M) \to H^{S^1}_{n-*}(\mathcal{L}L;\mathbb{K}). \tag{8.19}$$

This enables us to interpret a result of Davison [30] as providing obstructions to Lagrangian embeddings in Liouville manifolds with cyclic dilations, see Proposition 1.3.2.

*Proof of Proposition 1.3.2.* Let $M$ be a Liouville manifold with a cyclic dilation, and assume that there is an exact Lagrangian submanifold $L \subset M$ which is hyperbolic. It follows from Propositions 2.6.2 and 8.1.1 that there is a commutative diagram

$$\begin{tikzcd} SH^*_{S^1}(M) \arrow[r, "\mathbf{B}"] \arrow[d, "{[\widetilde{CL}]}"'] & SH^{*-1}(M) \arrow[d] \\ H^{S^1}_{n-*}(\mathcal{L}L;\mathbb{K}) \arrow[r, "\mathbf{B}"] & H_{n-*+1}(\mathcal{L}L;\mathbb{K}) \end{tikzcd} \tag{8.20}$$

where the vertical arrow on the right is the usual Viterbo map (8.1). By our assumption, there is a class $\tilde{b} \in SH^1_{S^1}(M)$ whose image under the Connes' map $\mathbf{B}$ is an invertible element $h \in SH^0(M)^\times$. By the commutativity of (8.20), and our assumption that $L$ is a $K(\pi,1)$ space, such a class induces an exact Calabi-Yau structure on the fundamental group algebra $\mathbb{K}[\pi_1(Q)]$, which contradicts the main result of [30]. □

It would also be interesting to take a look at the special case when $h = 1$ in the definition of a cyclic dilation, which leads to the following generalization of Corollary 6.3 of [103].

**Corollary 8.1.1.** *Suppose that the marking map* $\mathbf{B} : SH^1_{S^1}(M) \to SH^0(M)$ *hits the identity* $1 \in SH^0(M)$*, then* $M$ *cannot contain a closed exact Lagrangian submanifold* $L$ *which is a* $K(\pi,1)$ *space.*

*Proof.* Let $L \subset M$ be an exact Lagrangian submanifold which is topologically a $K(\pi,1)$ space. Since $T^*L$ is a Weinstein manifold, so Corollary 7.3.1 applies. It follows from Theorem 6.1.3 of [30] that the marking map $\mathbf{B} : SH^1_{S^1}(T^*L) \to SH^0(T^*L)$ cannot hit the identity. Suppose $M$ admits a cyclic dilation with $h = 1$, one can then use the commutative diagram (8.20) to get a contradiction. □

As an application, it follows from the existence result of [119] that the Milnor fibers $M_{a,\cdots,a}$ associated to the Brieskorn singularities

$$z_1^a + \cdots + z_{n+1}^a = 0, \tag{8.21}$$

where $n \geqslant a$, do not contain exact Lagrangian tori.

Corollary 8.1.1 can also be applied to deduce non-existence results concerning cyclic dilations. For example, consider the Weinstein 4-manifold $T_{1,1,0} \subset \mathbb{C}^3$ defined by the equation

$$x + y + xyz = 1, \tag{8.22}$$

which is the complement of a nodal elliptic curve $\Sigma \subset \mathbb{CP}^2$. This manifold is studied in Section 4.1 of [22], and it follows from the computation loc. cit that

$$SH^0(T_{1,1,0}) \cong \mathbb{K}[x, y, z]/(x + y + xyz - 1) \tag{8.23}$$

as $\mathbb{K}$-algebras. Since the polynomial $x + y + xyz - 1$ is irreducible over $\mathbb{K}$, the only invertible element in $SH^0(T_{1,1,0})$ is the identity. If $T_{1,1,0}$ admits a cyclic dilation, then $\mathbf{B} : SH^1_{S^1}(T_{1,1,0}) \to SH^0(T_{1,1,0})$ hits the identity. On the other hand, from the perspective of Legendrian surgery, $T_{1,1,0}$ can be constructed by attaching two 2-handles to the disc cotangent bundle $D^*T^2$, so there is an exact Lagranian torus $L \subset T_{1,1,0}$. Now Corollary 8.1.1 shows that $T_{1,1,0}$ does not admit a cyclic dilation.

One can attach one more 2-handle to $D^*T^2$ to get the Liouville domain associated to the affine surface $T_{1,1,1} \subset \mathbb{C}^3$ defined by the equation

$$x + y + z + xyz = 1. \tag{8.24}$$

Since it contains $\overline{T}_{1,1,0}$ as a Liouville subdomain, we conclude that $T_{1,1,1}$ does not admit a cyclic dilation. In fact, if $\overline{M}_0 \subset \overline{M}_1$ is a Liouville subdomain, there is a commutative diagram

$$\begin{tikzcd} SH^*_{S^1}(M_1) \arrow[r, "\mathbf{B}"] \arrow[d] & SH^{*-1}(M_1) \arrow[d] \\ SH^*_{S^1}(M_0) \arrow[r, "\mathbf{B}"] & SH^{*-1}(M_0) \end{tikzcd} \tag{8.25}$$

generalizing (8.20), from which we see that if $M_1$ admits a cyclic dilation, then so is $M_0$. The vertical map on the left of (8.25) is defined using a carefully chosen Hamiltonian, so that the Viterbo map on the chain level, which is a projection to the quotient complex, preserves the $S^1$-complex structures. We will not spell out the details here, as similar ideas will be developed in Section 9.2. Detailed construction can also be found in Appendix C of [117].

More generally, attaching 2-handles to $D^*T^2$ yields a sequence of Weinstein 4-manifolds $T_{p,q,r}$, with $p \geqslant q \geqslant r \geqslant 0$. When $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leqslant 1$, these are the Milnor fibers of parabolic and hyperbolic unimodal singularities mentioned in Section 1.2. Our discussions above imply the following:

**Proposition 8.1.2.** *The Weinstein manifold $T_{p,q,r}$ admits a cyclic dilation if and only if $q = r = 0$.*

*Proof.* Note that $T_{0,0,0}$ is symplectomorphic to $T^*T^2$, so it admits a quasi-dilation. $T_{1,0,0}$ is symplectomorphic to $\mathbb{C}^2 \setminus \{xy = 1\}$, it follows from Corollary 19.8 of [93] that there is a quasi-dilation in $SH^1(T_{1,0,0})$. Alternatively, one can compute its wrapped Fukaya category explicitly, whose endomorphism algebra turns out to be formal, and is quasi-isomorphic to the associative algebra $\mathbb{K}[x,y][(xy-1)^{-1}]$, whose superpotential description has been given in Example 2.2.1. The case when $p > 1$ can be argued similarly, since there are Lefschetz fibrations $T_{p,0,0} \to \mathbb{C}^*$ whose smooth fibers are $T^*S^1$. In fact, $T_{p,0,0}$ is symplectomorphic to the $\widetilde{A}_p$ plumbing of $T^*S^2$'s.

On the other hand, we have seen in the above that $T_{1,1,0}$ and $T_{1,1,1}$ do not admit cyclic dilations. Since any Weinstein manifold $T_{p,q,r}$ with $p \geqslant 1$ and $q \geqslant 1$ contains $\overline{T}_{1,1,0}$ as its Liouville subdomain, the non-existence of a cyclic dilation follows from the commutative diagram (8.25). $\square$

Observe that among the examples $T_{p,q,r}$ considered above, the existence of a cyclic dilation is in fact equivalent to the existence of a quasi-dilation. This is not surprising in view of Proposition 1.4.1. More interestingly, Proposition 8.1.2 implies the following:

**Corollary 8.1.2.** *Let $M$ be any 4-dimensional Milnor fiber associated to a non-simple singularity, then $M$ does not admit a cyclic dilation.*

*Proof.* This follows from the commutative diagram (8.25) and the adjacency of singularities ([60], Corollary 2.17). The latter implies the existence of some triple $(p, q, r)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, so that $M$ contains $\overline{T}_{p,q,r}$ as a Liouville subdomain. However, it follows from Proposition 8.1.2 that any such $T_{p,q,r}$ cannot admit a cyclic dilation. $\square$

# Chapter 9

# Existence of cyclic dilations

Let $M$ be a $2n$-dimensional Liouville manifold with $c_1(M) = 0$. In this chapter, we consider the existence and uniqueness questions concerning cyclic dilations. In Section 9.1, we use Koszul duality to show that the manifold $M_{3,3,3,3}$ admits a cyclic dilation. This example is non-trivial as $M_{3,3,3,3}$ does not admit a quasi-dilation. With the help of Lefschetz fibrations, one can produce infinitely many non-trivial examples starting from $M_{3,3,3,3}$. This is done in Section 9.2. Section 9.3 proves the uniqueness of smooth Calabi-Yau structures on the wrapped Fukaya categories of log general type affine varieties containing exact Lagrangian $K(\pi,1)$'s, from which Theorem 1.4.2 follows as a corollary. The discussions in Section 9.4 are mostly speculative, they are included here merely as supplements to Section 1.4.

## 9.1 Koszul duality, revisited

Although for the most part of Chapters 7 and 8 we have been taking a geometric viewpoint, dealing with cyclic dilations in equivariant symplectic cohomologies instead of exact Calabi-Yau structures on wrapped Fukaya categories, this section is an exception. Here we shall return to the original notion of an exact Calabi-Yau structure (Definition 1.3.1) which motivates the second part of this thesis, and study it essentially from the algebraic perspective, based on a result of Van den Bergh (cf. Theorem 9.1.1).

Before we proceed, first recall that smooth Calabi-Yau structures are Morita invariant, so it makes no difference to study smooth Calabi-Yau structures on an $A_\infty$-algebra $\mathcal{A}$ over some semisimple ring $\Bbbk$, or to consider them as Calabi-Yau structures on the $A_\infty$-category $\mathcal{A}^{perf}$. See Theorem 3.1 of [29] for an explanation of this fact.

One of the main ingredients of our proof of Theorem 1.4.1 is the following theorem due to Van den Bergh [111], which enables us to characterize a large class of exact Calabi-Yau $A_\infty$-algebras in terms of its Koszul dual.

**Theorem 9.1.1** ([111], Theorem 11.1)**.** *Let $\mathcal{A}$ be a homologically smooth, complete, augmented dg algebra over $\Bbbk$, so that $H^*(\mathcal{A})$ is concentrated in degrees $\leqslant 0$. Let $\mathcal{A}^!$ be the $\mathbb{Z}$-graded Koszul dual of $\mathcal{A}$. Then the following statements are equivalent:*

- *$\mathcal{A}^!$ is a proper $A_\infty$-algebra which, up to quasi-isomorphism, carries a cyclic $A_\infty$-structure of degree $n$.*
- *$\mathcal{A}$ is exact $n$-Calabi-Yau.*

Here, by *complete* we mean the underlying associative algebra of $\mathcal{A}$ is a quotient of the path algebra of some quiver completed at path length.

Theorem 9.1.1 should be understood in the more general framework of Koszul duality between Calabi-Yau structures, which we now describe. Recall that over a field $\mathbb{K}$ of characteristic 0, cyclic $A_\infty$-structures provide explicit models for the more general notion of a proper Calabi-Yau structure defined in Section 2.1. It is proved by Ganatra ([43], Theorem 2) that any full subcategory of the compact Fukaya category $\mathcal{F}(M)$ admits a geometrically defined proper Calabi-Yau structure. As a consequence, we have the following:

**Proposition 9.1.1** ([43], Corollary 2)**.** *Let $M$ be a Liouville manifold with $c_1(M) = 0$. If $\mathrm{char}(\mathbb{K}) = 0$, then any full $A_\infty$-subcategory of the compact Fukaya category $\mathcal{F}(M)$ is quasi-isomorphic to a cyclic $A_\infty$-category.*

If $\mathcal{A}$ and $\mathcal{B}$ are Koszul dual as $\mathbb{Z}$-graded $A_\infty$-algebras, then there is a duality

$$CH_{*-n}(\mathcal{A}) \cong \hom\left(CH_{*+n}(\mathcal{B}), \mathbb{K}\right) \tag{9.1}$$

between Hochschild chains, which suggests that under Koszul duality, non-degenerate cycles in $CH_{-n}(\mathcal{A})$ should correspond to maps

$$CH_{*+n}(\mathcal{B}) \to \mathbb{K} \tag{9.2}$$

which induce proper Calabi-Yau structures on $\mathcal{B}$. The theorem stated below is a slight variant of [29], Theorem 25.

**Theorem 9.1.2** (Cohen-Ganatra)**.** *Let $\mathcal{A}$ be a homologically smooth dg algebra over $\Bbbk$, and let $\mathcal{A}^!$ be a proper $A_\infty$-algebra so that $\mathcal{A}$ and $\mathcal{A}^!$ are Koszul dual as $\mathbb{Z}$-graded $A_\infty$-algebras. Then $\mathcal{A}$ carries a smooth Calabi-Yau structure if and only if $\mathcal{A}^!$ is a proper Calabi-Yau $A_\infty$-algebra.*

From this perspective, the content of Theorem 9.1.1 can be understood as saying that if we further impose the assumptions that $\mathcal{A}$ is complete and supported in non-positive degrees, then the Calabi-Yau structure on $\mathcal{A}$ induced by the proper Calabi-Yau structure on $\mathcal{A}^!$ is not only smooth, but also exact.

Geometrically, the $A_\infty$-Koszul duality between the endomorphism algebras of a set of generators in $\mathcal{F}(M)$ and $\mathcal{W}(M)$ has been studied in the first part of this thesis. However, to apply Theorem 9.1.1, we will need a result of Ekholm-Lekili [36], which shows that in certain cases Koszul duality between the Fukaya $A_\infty$-algebras $\mathcal{F}_M$ and $\mathcal{W}_M$ implies automatically the completeness of $\mathcal{W}_M$.

We use the set up and notations from Section 3.2. In particular, the Weinstein domain $\overline{M}$ is obtained by attaching $n$-handles to $\overline{M}_{-\Lambda}$ along the Legendrian submanifold $\Lambda \subset \partial\overline{M}_{-\Lambda}$. The augmentation (3.15) on the Chekanov-Eliashberg algebra $CE^*(\Lambda)$ allows us to write

$$CE^*(\Lambda) = \Omega LC_*(\Lambda), \tag{9.3}$$

where $\Omega$ is the Adams cobar construction, and $LC_*(\Lambda)$ is an $A_\infty$-coalgebra over $\Bbbk$, which is called the *Legendrian $A_\infty$-coalgebra* in [36], whose linear dual $LC_*(\Lambda)^\#$ is quasi-isomorphic to the Fukaya $A_\infty$-algebra $\mathcal{V}_M$ defined in Section 3.2. The *completed Chekanov-Eliashberg dg algebra* is defined to be

$$\widehat{CE}^*(\Lambda) := \mathrm{B}(\mathcal{V}_M)^\#, \tag{9.4}$$

where on the right-hand side the bar construction is taken with respect to the trivial augmentation $\varepsilon : \mathcal{V}_M \to \Bbbk$ defined by projecting to the idemponents in the degree 0 part.

To state the result from [36] which is crucial to our proof, we need a little bit more background on algebra. Denote by $\mathcal{C}$ any strictly counital $A_\infty$-coalgebra over $\Bbbk$ which carries a coaugmentation $\varepsilon^\vee : \Bbbk \to \mathcal{C}$. Denote by $\overline{\mathcal{C}} := \mathrm{coker}(\varepsilon^\vee)$ the coaugmentation ideal. There is a decreasing, exhaustive, bounded above filtration on the cobar complex

$$\Omega\mathcal{C} = F^0\Omega\mathcal{C} \supset F^1\Omega\mathcal{C} \supset \cdots, \tag{9.5}$$

where

$$F^p\Omega\mathcal{C} := \overline{\mathcal{C}}[-1]^{\otimes_{\Bbbk} p} \oplus \overline{\mathcal{C}}[-1]^{\otimes_{\Bbbk}(p+1)} \oplus \cdots . \tag{9.6}$$

The *completed cobar construction* is defined to be the inverse limit

$$\widehat{\Omega}\mathcal{C} := \varprojlim_p \Omega\mathcal{C}/F^p\Omega\mathcal{C}. \tag{9.7}$$

Taking $\mathcal{C} = LC_*(\Lambda)$ to be the Legendrian $A_\infty$-coalgebra, we can write

$$CE^*(\Lambda) = \Bbbk \oplus \bigoplus_{i=1}^{\infty} \overline{LC}_*(\Lambda)[-1]^{\otimes_{\Bbbk} i}, \tag{9.8}$$

where $\overline{LC}_*(\Lambda) \subset LC_*(\Lambda)$ is the submodule without the idempotents. It follows from the definition (9.4) that as an algebra over $\Bbbk$,

$$\widehat{CE}^*(\Lambda) = \Bbbk\langle\langle \overline{LC}_*(\Lambda) \rangle\rangle, \tag{9.9}$$

which is the completed tensor algebra of $\mathbb{K}\langle\mathcal{R}\rangle$, regarded as a module over $\Bbbk$. In particular, there is a completion map

$$\phi : CE^*(\Lambda) \to \widehat{CE}^*(\Lambda). \tag{9.10}$$

The following theorem follows from Proposition 18 of [36], which enables us to recover part of the information of the wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_M$ from the compact Fukaya $A_\infty$-algebra $\mathcal{F}_M$ even when they are not necessarily Koszul dual.

**Theorem 9.1.3** (Ekholm-Lekili)**.** *Suppose that the Weinstein domain $\overline{M}_{-\Lambda}$ is subcritical, and the Floer cohomology algebra $H^*(\mathcal{V}_M)$ is non-negatively graded with $H^0(\mathcal{V}_M) \cong \Bbbk$, then there is a quasi-isomorphism*

$$\widehat{CE}^*(\Lambda) \simeq \widehat{\Omega}\mathcal{V}_M^{\#}. \tag{9.11}$$

*In other words, $\widehat{\Omega}\mathcal{V}_M^{\#}$ and $\mathcal{V}_M$ are Koszul dual as $A_\infty$-algebras.*

We now apply Theorem 9.1.1 to concrete geometric situations. As a quick application, let $T$ be a tree with vertex set $T_0$. For each $v \in T_0$ we associate a simply-connected closed manifold $Q_v$ of dimension $n \geqslant 3$. For simplicity, we also assume that $Q_v$ is *Spin*. Denote by $M_T$ the Weinstein manifold obtained by plumbing the cotangent bundles $T^*Q_v$ according to the tree $T$.

**Proposition 9.1.2.** *The wrapped Fukaya category $\mathcal{W}(M_T)$ carries an exact Calabi-Yau structure.*

*Proof.* It follows from the proof of Theorem 54 of [36] that the wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_{M_T}$ is quasi-isomorphic to a dg algebra concentrated in degrees $\leqslant 0$, and it is Koszul dual to the Fukaya $A_\infty$-algebra $\mathcal{F}_{M_T}$ of the compact cores $\{Q_v\}_{v \in T_0}$. Moreover, combining Theorem 9.1.3 above and Theorem 54 of [36] one finds that $\mathcal{W}_M$ is quasi-isomorphic to its completion. Since $\mathrm{char}(\mathbb{K}) = 0$, Proposition 9.1.1 implies that up to quasi-isomorphism, $\mathcal{F}_{M_T}$ carries a cyclic $A_\infty$-structure. Now the conclusion follows from Theorem 9.1.1. □

Note that this gives an alternative way of seeing that the cotangent bundle $T^*Q$ of a simply-connected manifold $Q$ admits a cyclic dilation, compare with our discussions at the end of Section 7.2. It is an interesting question whether the Weinstein manifolds $M_T$ admit higher dilations.

Our second application deals with the specific case of the affine hypersurface $M_{3,3,3,3} \subset \mathbb{C}^4$. Recall that the Liouville 6-manifold $M_{3,3,3,3}$ arises as the Milnor fiber associated to the isolated singularity at the origin

$$x^3 + y^3 + z^3 + w^3 = 0, \tag{9.12}$$

which is known as a *3-fold triple point*. The smoothing of this singularity has been studied by Smith-Thomas in [106], according which we know that there is a basis of

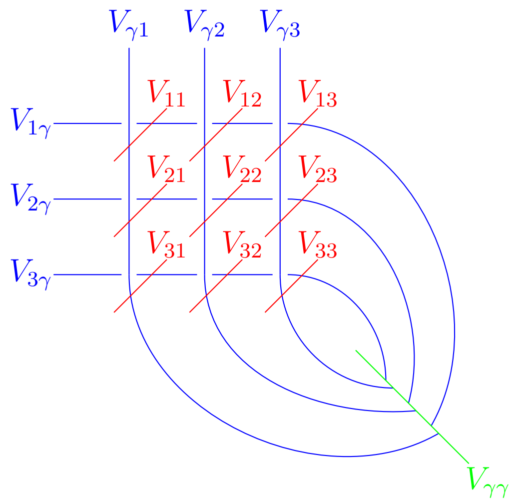


Figure 9.1: Configuration of vanishing cycles in $M_{3,3,3,3}$, note that the spheres coloured in blue are mutually disjoint

vanishing cycles in $M_{3,3,3,3}$ which consists of a configuration of 16 Lagrangian spheres, whose intersection pattern is indicated in Figure 9.1, where each arc represents a Lagrangian sphere.

One obtains from this the Legendrian surgery description of $M_{3,3,3,3}$:

**Lemma 9.1.1.** *The Milnor fiber $M_{3,3,3,3}$ is the result of attaching Weinstein 3-handles to $D^6$ along a Legendrian surface $\Lambda_{3,3,3,3} \subset (S^5, \xi_{std})$, which is a disjoint union of 16 standard unknotted Legendrian $S^2$'s. Up to Legendrian isotopy, the Legendrian fronts of 10 of the components in link $\Lambda_{3,3,3,3}$ are depicted in Figure 9.3, where each Legendrian unknot in the picture should be understood as a 2-sphere obtained by spinning the 1-dimensional unknot around along the vertical axis of symmetry of its front projection. The remaining 6 components $\Lambda_{12}, \Lambda_{13}, \Lambda_{21}, \Lambda_{23}, \Lambda_{31}, \Lambda_{32}$ are unknots linking $\Lambda_{1\gamma}$ and $\Lambda_{\gamma 2}$, $\Lambda_{1\gamma}$ and $\Lambda_{\gamma 3}$, $\Lambda_{2\gamma}$ and $\Lambda_{\gamma 1}$, $\Lambda_{2\gamma}$ and $\Lambda_{\gamma 3}$, $\Lambda_{3\gamma}$ and $\Lambda_{\gamma 1}$, and $\Lambda_{3\gamma}$ and $\Lambda_{\gamma 2}$ respectively, with all the linking numbers being $\pm 1$. They are pairwise* *disjoint and disjoint from $\Lambda_{\gamma\gamma}$.*

*Proof.* Consider the Lefschetz fibration $t : \mathbb{C}^3 \to \mathbb{C}$ obtained as the Morsification of the polynomial $x^3+y^3+z^3$. The smooth fiber of $t$ is symplectomotphic to the Milnor fiber $T_{3,3,3}$ associated to the singularity $x^3+y^3+z^3=0$, and its total monodromy is the composition of Dehn twists along a basis of 8 vanishing cycles in $T_{3,3,3}$, see Section 4.2 of [60] for a detailed description of this Lefschetz fibration. By Theorem 4.4 of [112], this describes $D^6$ as the result of attaching 8 Weinstein 3-handles to $T_{3,3,3} \times D^2$ along a link of 8 Legendrian 2-spheres in $T_{3,3,3} \times S^1$, which restricts to the basis of vanishing cycles in $T_{3,3,3}$. Moreover, $M_{3,3,3,3}$ also carries a Lefschetz fibration $\pi : M_{3,3,3,3} \to \mathbb{C}$, with $T_{3,3,3}$ as its smooth fiber, under which the vanishing cycles of $M_{3,3,3,3}$ described in Figure 9.1 can be realized as Lagrangian matching spheres. Figure 9.2 gives a description of the base of $\pi$, where the 24 crosses are critical values, which are divided into three groups, and $\pi^{-1}(\star)$ is a smooth fiber. See Section 2.5 of [60], where the detailed construction of such a Lefschetz fibration is explained. The blue curve in Figure 9.2 which connects two different critical values of $\pi$ is the projection of a matching sphere $V \subset M_{3,3,3,3}$. This shows that $M_{3,3,3,3}$ can be constructed by attaching 24 Weinstein 3-handles to $T_{3,3,3} \times D^2$ along a link of 24 Legendrian $S^2$'s, whose restrictions in $\pi^{-1}(\star)$ are vanishing cycles of $\pi$. Note that it can be arranged so that the basis of vanishing cycles of $\pi$ contains the aforementioned basis of vanishing cycles of $t$ as a subset. Comparing with the handlebody decomposition of $D^6$ described above, this realizes the Weinstein domain $\overline{M}_{3,3,3,3}$ as $D^6$ with 16 Weinstein 3-handles attached along a link of 16 unknotted Legendrian $S^2$'s in $(S^5, \xi_{std})$. When restricting to the smooth fiber $\pi^{-1}(\star)$ of $\pi$, these Legendrian spheres form a subset of the basis of vanishing cycles of $\pi$, and every one of them lies in a matching sphere, in the fiber above $\star \in \mathbb{C}$. More precisely, consider the associated Lefschetz fibration $\bar{\pi} : \overline{M}_{3,3,3,3} \to D^2$ on the Liouville domain (with corners) $\overline{M}_{3,3,3,3}$ obtained by cutting off the cylindrical ends of the fibers, and removing the preimage of the part outside of the dashed circle in Figure 9.2, where the fibration is locally trivial (this is the original set up of [89]). If we cut the base of $\pi$ along the orange dashed arc in Figure 9.2, the preimage under $\bar{\pi}$ of the lower left half of the disc, which we denote by $D_-$, with the corners rounded-off, is deformation equivalent to $D^6$, and the restrictions of the 16 matching spheres to $\bar{\pi}^{-1}(D_-)$ become exact Lagrangian fillings of the corresponding vanishing cycles, which are considered as Legendrian spheres in the contact boundary $\partial\bar{\pi}^{-1}(D_-)$. For example, for the matching sphere $V$ in the figure, its restriction $\overline{V} := V \cap \bar{\pi}^{-1}(D_-)$ is a Lagrangian disc, which fills its boundary $\partial\overline{V} \subset \partial\bar{\pi}^{-1}(D_-)$, which is a Legendrian 2-sphere. In this way, the linking pattern of the 16 Legendrian $S^2$'s in $\partial\bar{\pi}^{-1}(D_-)$ is determined by the intersection pattern of the Lagrangian matching spheres in $M_{3,3,3,3}$, which is shown in Figure 9.1.

□

In Figure 9.3, we have arranged so that the labellings of the components of $\Lambda_{3,3,3,3}$ coincide with that of the vanishing cycles in Figure 9.1, which means that for any vanishing cycle $V_{\bullet\bullet}$, the Lagrangian 3-disc $V_{\bullet\bullet} \cap D^6$ is a filling of the component $\Lambda_{\bullet\bullet} \subset \Lambda_{3,3,3,3}$ with the same labelling. The set of Lagrangian cocores $\{L_{\bullet\bullet}\}$ will be labelled in the same way, with $L_{\bullet\bullet}$ being the cocore of the 3-handle attached along $\Lambda_{\bullet\bullet}$. As a consequence, $V_{\bullet\bullet}$ intersects $L_{\bullet\bullet}$ non-trivially and transversely at a unique point if and only if they have the same labelling. We denote by $\mathcal{W}_{M_{3,3,3,3}}$ the Fukaya $A_\infty$-algebra of the cocores $\{L_{\bullet\bullet}\}$.

**Lemma 9.1.2.** *Up to quasi-isomorphism, the wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_{M_{3,3,3,3}}$ is concentrated in degrees $\leqslant 0$.*

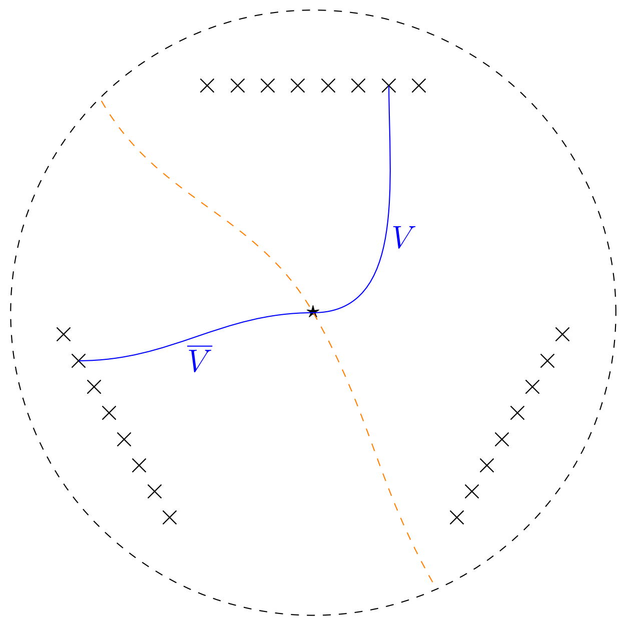


Figure 9.2: Base of the Lefschetz fibration $\pi : M_{3,3,3,3} \to \mathbb{C}$

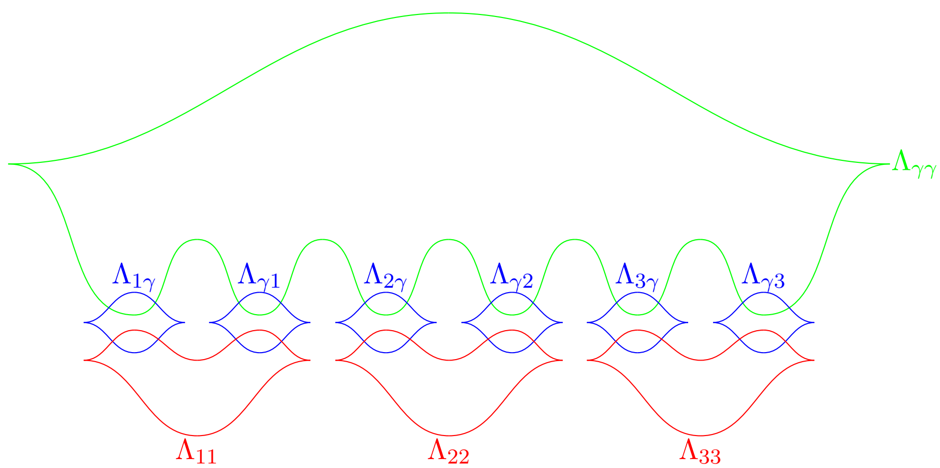


Figure 9.3: Front view of the Legendrian front of $\Lambda_{3,3,3,3}$, where the components $\Lambda_{12}, \Lambda_{13}, \Lambda_{21}, \Lambda_{23}, \Lambda_{31}, \Lambda_{32}$ are omitted since they are covered by the other components

*Proof.* According to [16, 33, 36], the surgery map

$$\Theta : \mathcal{W}_{M_{3,3,3,3}} \to CE^*(\Lambda_{3,3,3,3}) \tag{9.13}$$

induces an isomorphism on homologies, so it suffices to check that the Chekanov-Eliashberg dg algebra $CE^*(\Lambda_{3,3,3,3})$ is concentrated in non-positive degrees up to quasi-isomorphism.

Equip $\Lambda_{3,3,3,3}$ with a Maslov potential $\mu : \Lambda_{3,3,3,3} \to \mathbb{Z}$ as follows. For the green component $\Lambda_{\gamma\gamma}$, let $\mu = 1$ on the upper strand, and $\mu = 0$ on the lower strand. For each blue component, put $\mu = 0$ on the upper strand, and $\mu = -1$ on the lower strand. Finally, for each red component, set $\mu = -1$ on the upper strand, and $\mu = -2$ on the lower strand.

To show that $CE^*(\Lambda_{3,3,3,3})$ is quasi-isomorphic to a dg algebra with all the generators concentrated in non-positive degrees, we make use of the cellular model $\mathcal{C}(\Lambda_{3,3,3,3})$ of $CE^*(\Lambda_{3,3,3,3})$ recalled in Chapter 5. Observe that by our choice of the Maslov potential $\mu$, for any sheet $S_m$ with larger height than $S_n$, we necessarily have $\mu(S_m) \geqslant \mu(S_n)$, which according to (5.8) implies that all the generators of the form $b_\alpha^{m,n}$ and $c_\alpha^{m,n}$ have non-positive gradings. However, there may still be generators $a_\alpha^{m,n}$ with $|a_\alpha^{m,n}| = 1$. For each such generator, consider any 1-cell $e_\beta^1$ in the $\Lambda_{3,3,3,3}$-compatible polygonal decomposition which ends at $e_\alpha^0$, with the other end point being $e_\gamma^0$ for some $\gamma$. By definition of the differential $d_\mathcal{C}$ we have (over $\mathbb{K} = \mathbb{Z}/2$)

$$d_\mathcal{C} b_\beta^{m,n} = a_\alpha^{m,n} + a_\gamma^{m,n} + \sum_{m<k<n} a_\alpha^{m,k} b_\beta^{k,n} + \sum_{m<k<n} b_\beta^{m,k} a_\gamma^{k,n}, \tag{9.14}$$

which shows that there is a quasi-isomorphism

$$\mathcal{C}(\Lambda_{3,3,3,3}) \cong \mathcal{C}(\Lambda_{3,3,3,3}) / \left\langle d_\mathcal{C} b_\beta^{m,n}, b_\beta^{m,n} \right\rangle \tag{9.15}$$

between $\mathcal{C}(\Lambda_{3,3,3,3})$ and its quotient dg algebra, where $\left\langle d_\mathcal{C} b_\beta^{m,n}, b_\beta^{m,n} \right\rangle$ is the ideal generated by $d_\mathcal{C} b_\beta^{m,n}$ and $b_\beta^{m,n}$. For any relevant 0-cell $e_\alpha^0$, it is always possible to find a corresponding 1-cell $e^1_\beta$ in the cellular decomposition, so that $a^{m,n}_\gamma = 0$ for the 0-cell $e^0_\gamma$. This can be checked directly using the definition of the cellular dg algebra, or one may observe that geometrically, all the Reeb chords ending on $\Lambda_{3,3,3,3}$ correspond to generators of the form $b^{m,n}_\alpha$ or $c^{m,n}_\alpha$. In other words, by replacing the cellular dg algebra $\mathcal{C}(\Lambda_{3,3,3,3})$ with a quasi-isomorphic dg algebra if necessary, all the generators with positive degrees can be cancelled out.

Since the Legendrian front of $\Lambda_{3,3,3,3}$ does not involve any swallowtail singularity, as explained in the proof of Lemma A.2.1 below, the cancellation of the generators does not depend on the ground field $\mathbb{K}$, so we conclude that $CE^*(\Lambda_{3,3,3,3})$ is quasi-isomorphic to a dg algebra with all the generators concentrated in degrees $\leqslant 0$ over any field $\mathbb{K}$. $\square$

Since $\subset M_{3,3,3,3}$ is the Milnor fiber associated to a (weighted) homogeneous singularity, it follows from Lemmas 4.15 and 4.16 of [97] that

$$\left(\tau_{V_{11}} \circ \cdots \circ \tau_{V_{33}} \circ \tau_{V_{\gamma 1}} \circ \tau_{V_{\gamma 2}} \circ \tau_{V_{\gamma 3}} \circ \tau_{V_{1\gamma}} \circ \tau_{V_{2\gamma}} \circ \tau_{V_{3\gamma}} \circ \tau_{V_{\gamma\gamma}}\right)^3 = [-2]. \tag{9.16}$$

Since the right hand side of (9.16) is a non-trivial degree shift, by Seidel's long exact sequence [98], the compact Fukaya category $\mathcal{F}(M_{3,3,3,3})$ is split-generated by the 16 Lagrangian spheres

$$V_{11}, \cdots, V_{33}, V_{\gamma 1}, V_{\gamma 2}, V_{\gamma 3}, V_{1\gamma}, V_{2\gamma}, V_{3\gamma}, V_{\gamma\gamma}. \tag{9.17}$$

In particular, denote by $\mathcal{F}_{M_{3,3,3,3}}$ the Fukaya $A_\infty$-algebra of these vanishing cycles, then there are quasi-equivalences

$$\mathcal{F}^{perf}_{M_{3,3,3,3}} \cong \mathcal{F}(M_{3,3,3,3})^{perf}, \mathcal{F}^{prop}_{M_{3,3,3,3}} \cong \mathcal{F}(M_{3,3,3,3})^{prop}, \tag{9.18}$$

where by $\mathcal{A}^{prop}$ we mean the $A_\infty$-category of proper modules over an $A_\infty$-algebra or $A_\infty$-category $\mathcal{A}$. As a Corollary to Lemma 9.1.2, we conclude the following:

**Corollary 9.1.1.** *The Lagrangian submanifolds* $\{V_{\bullet\bullet}\}$ *admit gradings for which the* $A_\infty$*-algebra* $\mathcal{F}_{M_{3,3,3,3}}$ *is concentrated in degrees* $0 \leqslant * \leqslant 3$*, and its degree 0 part is isomorphic to* $\Bbbk := \bigoplus_{i=1}^{16} \mathbb{K}e_i$*.*

*Proof.* Since we have seen in Lemma 9.1.2 that the Chekanov-Eliashberg dg algebra $CE^*(\Lambda_{3,3,3,3})$ is non-positively graded up to quasi-isomorphism, this is a consequence of the Eilenberg-Moore equivalence (3.16). □

**Remark 9.1.1.** *Compare with Lemma 4.5 of* [*105*]*, where a similar result is proved for the WKB algebra of Lagrangian spheres. We expect that the same grading property holds for the Fukaya* $A_\infty$*-algebra of a basis of vanishing cycles in any Milnor fiber* $M_{a_1,\cdots,a_{n+1}}$ *with* $\sum_{i=1}^{n+1} \frac{1}{a_i} > 1$*.*

The following lemma is essentially proved in [72], and we only give a brief sketch of their argument.

**Lemma 9.1.3** (Lekili-Ueda)**.** *There is a fully faithful embedding*

$$\mathcal{W}(M_{3,3,3,3}) \hookrightarrow \mathcal{F}(M_{3,3,3,3})^{mod}, \tag{9.19}$$

*where* $\mathcal{F}(M_{3,3,3,3})^{mod}$ *is the category of* $A_\infty$*-modules over* $\mathcal{F}(M_{3,3,3,3})$*.*

*Proof.* Since the right-hand side of (9.16) is a non-trivial degree shift, the diagonal bimodule over $\mathcal{W}(M_{3,3,3,3})$ is a colimit of $\mathcal{F}(M_{3,3,3,3}) \otimes \mathcal{F}(M_{3,3,3,3})^{op}$ in the category of $\mathcal{W}(M_{3,3,3,3})$-bimodules, see Proposition 7.7 of [72]. The diagonal argument of Beilinson [12] then implies that $\mathcal{W}(M_{3,3,3,3})$ is a subcategory of $\mathcal{F}(M_{3,3,3,3})^{mod}$. □

As a corollary, there is an isomorphism

$$HH^*(\mathcal{F}(M_{3,3,3,3})) \cong HH^*(\mathcal{W}(M_{3,3,3,3})) \tag{9.20}$$

between Hochschild cohomologies. In fact, it is possible to strengthen Lemma 9.1.3 by showing that $\mathcal{F}_{M_{3,3,3,3}}$ and $\mathcal{W}_{M_{3,3,3,3}}$ are Koszul dual as $A_\infty$-algebras. To do this, we need some more background from algebra.

Let $\mathcal{D}$ be a triangulated category, and let $P$ be an object of $\mathcal{D}$. We say that $P$ is a *classical generator* of $\mathcal{D}$ if $P$ split-generates $\mathcal{D}$, and $P$ is a *weak generator*, or it weakly generates $\mathcal{D}$, if for any non-zero object $Q$ of $\mathcal{D}$, we have

$$\hom_{\mathcal{D}}(P, Q[j]) \neq 0 \tag{9.21}$$

for some $j \in \mathbb{Z}$. Note that these definitions extend straightforwardly to the case of a family of objects.

The notion of a classical generator is useful as it allows us to recover the information of the whole category by expressing an arbitrary object $X$ of $\mathcal{D}$ as a direct summand of some iterated mapping cone of shifted copies of the generator $P$, which is in general not the case of a weak generator. In order to establish an analogous picture for weak generators, we need to impose an intrinsic condition called compactness.

**Definition 9.1.1.** *Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A compact object $K$ of $\mathcal{D}$ is an object such that the map*

$$\bigoplus_{i\in I} \hom_{\mathcal{D}}(K, P_i) \to \hom_{\mathcal{D}}\left(K, \bigoplus_{i\in I} P_i\right) \tag{9.22}$$

*is bijective for any set $I$ and objects $\{P_i\}_{i\in I}$. The compact objects of $\mathcal{D}$ form a full triangulated subcategory, which will be denoted by $\mathcal{D}_c$.*

Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. We say that $\mathcal{D}$ is *compactly generated* if there exists a set $\{K_i\}_{i\in I}$ of compact objects such that $\bigoplus_{i\in I} K_i$ weakly generates $\mathcal{D}$. It turns out that any object of a compactly generated triangulated category can be recovered from its compact weak generators as a homotopy colimit (cf. Lemma 13.34.3 of [107]). As a consequence, we have the following:

**Proposition 9.1.3** ([78], Theorem 2.2). *Let $\mathcal{D}$ be a compactly generated triangulated category with a set of compact weak generators $\{K_i\}_{i\in I}$, then $\mathcal{D}$ is the smallest strictly full triangulated subcategory of $\mathcal{D}$ which contains $\{K_i\}_{i\in I}$ and is closed under direct sums.*

For compactly generated triangulated categories, we have the following fact which relates classical generators of $\mathcal{D}_c$ to weak generators of $\mathcal{D}$:

**Proposition 9.1.4** ([107], Proposition 13.34.6). *Let $\mathcal{D}$ be a compactly generated triangulated category, and $K$ is a compact object of $\mathcal{D}$. Then $K$ is a weak generator of $\mathcal{D}$ if and only if it is a classical generator of $\mathcal{D}_c$.*

In practice, we will be dealing with triangulated $A_\infty$-categories in the sense of [89]. Given such a category, the results above can be applied to the associated derived category, which is a genuine triangulated category.

**Proposition 9.1.5.** *There are quasi-isomorphisms between $\mathbb{Z}$-graded $A_\infty$-algebras over $\mathbb{k}$:*

$$R\mathrm{Hom}_{\mathcal{F}_{M_{3,3,3,3}}}(\mathbb{k},\mathbb{k}) \cong \mathcal{W}_{M_{3,3,3,3}}, R\mathrm{Hom}_{\mathcal{W}_{M_{3,3,3,3}}}(\mathbb{k},\mathbb{k}) \cong \mathcal{F}_{M_{3,3,3,3}}. \tag{9.23}$$

*Proof.* Consider the composition of $A_\infty$-functors

$$\mathcal{W}^{perf}_{M_{3,3,3,3}} \xrightarrow{\mathcal{I}} \mathcal{F}^{prop}_{M_{3,3,3,3}} \xrightarrow{\mathcal{K}} \left(\widehat{\Omega}\mathcal{F}^{\#}_{M_{3,3,3,3}}\right)^{perf}, \tag{9.24}$$

where the functor $\mathcal{I}$ comes from the fully faithful embedding

$$\mathcal{W}(M_{3,3,3,3})^{perf} \hookrightarrow \mathcal{F}(M_{3,3,3,3})^{prop}, \tag{9.25}$$

whose existence relies on the fully faithful embedding $\mathcal{W}(M_{3,3,3,3}) \hookrightarrow \mathcal{F}(M_{3,3,3,3})^{mod}$ established in Lemma 9.1.3, and the fact that the wrapped Floer cohomology $HW^*(L,K)$

is finite dimensional for any object $L$ of the wrapped Fukaya category, as long as $K \subset M_{3,3,3,3}$ is a compact Lagrangian submanifold, while

$$\mathcal{K} := R\mathrm{Hom}_{\mathcal{F}_{M_{3,3,3,3}}}(\cdot, \Bbbk) \tag{9.26}$$

is the Koszul duality functor considered in Section 3.4. By Corollary 9.1.1, Theorem 9.1.3 applies and shows that the $A_\infty$-algebras $\mathcal{F}_{M_{3,3,3,3}}$ and $\widehat{\Omega}\mathcal{F}^{\#}_{M_{3,3,3,3}}$ are Koszul dual, so the Koszul duality functor $\mathcal{K}$ is a quasi-equivalence, see [64]. In particular, the composition $\mathcal{K} \circ \mathcal{I}$ defines a fully faithful embedding.

We now show that $\mathcal{K} \circ \mathcal{I}$ is actually a quasi-equivalence. Since the vanishing cycles $\{V_{\bullet\bullet}\}$ in (9.17) can be identified with proper modules over the wrapped Fukaya category $\mathcal{W}(M_{3,3,3,3})$, they define a set of compact objects of $D^{perf}(\mathcal{W}_{M_{3,3,3,3}})$. Under the derived functor $D\mathcal{I}$ associated to the fully faithful embedding $\mathcal{I}$, the Yoneda modules $\{\mathcal{Y}_{V_{\bullet\bullet}}\}$ (regarded as objects of $\mathcal{F}^{perf}_{M_{3,3,3,3}}$) associated to the vanishing cycles become compact weak generators of $D^{prop}(\mathcal{F}_{M_{3,3,3,3}})$, the derived category of proper modules over $\mathcal{F}_{M_{3,3,3,3}}$. This is a consequence of Proposition 9.1.4, as $D^{prop}(\mathcal{F}_{M_{3,3,3,3}})$ is compactly generated by the subcategory $D^{perf}(\mathcal{F}_{M_{3,3,3,3}})$, and $\{\mathcal{Y}_{V_{\bullet\bullet}}\}$ split-generate $\mathcal{F}^{perf}_{M_{3,3,3,3}}$. By Proposition 9.1.3, there is then a fully faithful embedding

$$D^{prop}(\mathcal{F}_{M_{3,3,3,3}}) \hookrightarrow D^{perf}(\mathcal{W}_{M_{3,3,3,3}}). \tag{9.27}$$

This shows that $\mathcal{I}$ is actually a quasi-equivalence, so is the composition $\mathcal{K} \circ \mathcal{I}$.

We have proved the existence of a quasi-equivalence

$$\mathcal{K} \circ \mathcal{I} : \mathcal{W}^{perf}_{M_{3,3,3,3}} \xrightarrow{\simeq} \left(\widehat{\Omega}\mathcal{F}^{\#}_{M_{3,3,3,3}}\right)^{perf}. \tag{9.28}$$

Taking the direct sums of the generators on both sides yields a quasi-isomorphism between the respective endomorphism algebras $\mathcal{W}_{M_{3,3,3,3}} \cong \widehat{\Omega}\mathcal{F}^{\#}_{M_{3,3,3,3}}$, from which the Koszul duality between $\mathcal{F}_{M_{3,3,3,3}}$ and $\mathcal{W}_{M_{3,3,3,3}}$ follows. $\square$

*Proof of Theorem 1.4.1.* Denote by $\mathcal{F}_{M_{3,3,3,3}}$ the Fukaya $A_\infty$-algebra of the basis of vanishing cycles $\{V_{\bullet\bullet}\}$ depicted in Figure 9.1. It follows from Proposition 9.1.1 that $\mathcal{F}_{M_{3,3,3,3}}$ is a cyclic $A_\infty$-algebra. Combining Theorem 9.1.3 and Proposition 9.1.5 we see that $\mathcal{W}_{M_{3,3,3,3}}$ is quasi-isomorphic to a complete dg algebra, whose Koszul dual is $\mathcal{F}_{M_{3,3,3,3}}$. Moreover, Lemma 9.1.2 implies that $H^*\left(\mathcal{W}_{M_{3,3,3,3}}\right)$ is supported in non-positive degrees. Applying Theorem 9.1.1 completes the proof. □

As a by-product, we have the following non-formality result. A similar result is proved in Theorem 8.3 of [72].

**Corollary 9.1.2.** *The Fukaya $A_\infty$-algebra $\mathcal{V}_{M_{3,3,3,3}}$ is not formal over $\mathbb{k}$.*

*Proof.* Suppose that $\mathcal{V}_{M_{3,3,3,3}}$ is formal, then the Euler vector field defines a non-trivial class $[eu_{\mathcal{F}}] \in HH^1(\mathcal{F}(M_{3,3,3,3}))$. By Lemma 9.1.3, there is an isomorphism $HH^*(\mathcal{F}(M_{3,3,3,3})) \cong HH^*(\mathcal{W}(M_{3,3,3,3}))$ (as Gerstenhaber algebras, but may not be compatible with the BV structures on both sides), under which $[eu_{\mathcal{F}}]$ goes to a non-trivial class $b \in HH^1(\mathcal{W}(M_{3,3,3,3}))$, which induces via the composition

$$HH_{-n+1}(\mathcal{W}(M_{3,3,3,3})) \xrightarrow{\mathbb{I}} HC_{-n+1}(\mathcal{W}(M_{3,3,3,3})) \xrightarrow{\mathbb{B}} HH_{-n}(\mathcal{W}(M_{3,3,3,3})) \tag{9.29}$$

a weak smooth Calabi-Yau structure on $\mathcal{W}(M_{3,3,3,3})$, see Section 3.5, where it is proved that $\Delta_{CY}\left(\frac{1}{3}b\right) = 1$, with $\Delta_{CY}$ being the BV operator on $HH^*(\mathcal{W}(M_{3,3,3,3}))$ induced by some smooth Calabi-Yau structure on $\mathcal{W}(M_{3,3,3,3})$. Under the BV algebra isomorphism

$$HH^*(\mathcal{W}(M_{3,3,3,3})) \cong SH^*(M_{3,3,3,3}) \tag{9.30}$$

established by Ganatra, $b$ induces a quasi-dilation in $SH^1(M_{3,3,3,3})$, since changing the smooth Calabi-Yau structure on $\mathcal{W}(M_{3,3,3,3})$ has the effect of transforming the original BV operator $\Delta_{CY}$ to another one $h^{-1}\Delta_{CY}h$, for some $h \in HH^0(\mathcal{W}(M_{3,3,3,3}))^\times$.

To complete the proof, we follow the argument of Example 2.7 of [96] to show that $M_{3,3,3,3}$ does not admit a quasi-dilation. Consider the Milnor fiber $M$ of a 5-fold triple point, using the Morse-Bott spectral sequence [90] which converges to $SH^*(M)$, one can deduce that $SH^1(M) = 0$, which in particular implies that $M$ does not admit a quasi-dilation. On the other hand, there is a Lefschetz fibration $M \to \mathbb{C}$ on $M$ whose smooth fiber $F$ is symplectomorphic to the Milnor fiber of a 4-fold triple point. By Lemma 19.5 of [93], it follows that there is no quasi-dilation in $SH^1(F)$. Similarly, $F$ also admits a Lefschetz fibration $F \to \mathbb{C}$ with the Milnor fiber $M_{3,3,3,3}$ as its fiber, from which we conclude that there is no quasi-dilation in $SH^1(M_{3,3,3,3})$. $\square$

## 9.2 Lefschetz fibrations

This section is devoted to the proof of Theorem 1.3.1, which allows us to get new examples Liouville manifolds which admit cyclic dilations in terms of known ones. The argument here is a slight variation of those in Section 7 of [103] and Section 5.3 of [117].

We use the general set up of Section 7 of [103]. Let $\pi : M \to \mathbb{C}$ be an exact symplectic Lefschetz fibration, which means that its smooth fibers $F$ are completions of Liouville doomains $\overline{F}$. More explicitly, we require that

- For some almost complex structure $J \in \mathcal{J}(M)$, the map $\pi$ is $(J, j)$-holomorphic, where $j$ is the standard complex structure on $\mathbb{C}$.
- $\pi$ has finitely many isolated critical points, so that each singular fiber contains at most one critical point, and the almost complex structure $J$ is locally integrable near each of these critical points.

- There is a relatively open compact subset $\overline{M} \subset M$, so that its complement $M \backslash \overline{M}$ is identified with

$$\widetilde{M} := (\mathbb{R}^+ \times T) \cup_{\mathbb{R}^+ \times S^1 \times \mathbb{R}^+ \times \partial \overline{F}} (\mathbb{C} \times \mathbb{R}^+ \times \partial \overline{F}), \tag{9.31}$$

  where

$$T = (\mathbb{R} \times F)/(t, x) \sim (t-1, \mu(x)) \tag{9.32}$$

  is the mapping torus, with $\mu$ being the total monodromy of $\pi$. By construction, $\overline{M}$ is a manifold with corners, which coincides with the Liouville domain associated to $M$ up to deformation once the corners are rounded off.

- Fix the choice of a trivialization of the canonical bundle $K_M$, which induces a trivialization of $K_F$, the canonical bundle of the fiber.

Given such a Lefschetz fibration, consider the autonomous Hamiltonian $H_M : M \to \mathbb{R}$ defined by

$$H_M = H_F + \pi^* H_{\mathbb{C}}, \tag{9.33}$$

where $H_{\mathbb{C}}(z) = \varepsilon |z - c|^2/2$ for some $\varepsilon > 0$ is a function on the base, and $H_F : F \to \mathbb{R}$ is a Hamiltonian on the fiber which is linear on the cylindrical end $[1, \infty) \times \partial \overline{F}$ with slope $\lambda > 0$, where $\lambda \notin \mathcal{P}_F$. By Lemma 7.2 of [103], for sufficiently small $\varepsilon$, there is a short exact sequence

$$0 \to \mathbb{K}^{\mathrm{Crit}(\pi)}[-n] \to CF^*_{vert}(M, \lambda) \to CF^*(F, \lambda) \to 0, \tag{9.34}$$

where the Floer complexes $CF^*_{vert}(M, \lambda)$ and $CF^*(F, \lambda)$ are defined by choosing time-dependent perturbations of the autonomous Hamiltonians $H_M$ and $H_F$, and the notation $\mathbb{K}^{\mathrm{Crit}(\pi)}[-n]$ means the complex with trivial differential so that there is a copy of $\mathbb{K}$ in degree $n$ for every critical point of $\pi$. Here, we use the notation

$CF^*_{vert}(M,\lambda)$ to indicate that when $\lambda \to \infty$, the slope of our Hamiltonian $H_M$ only increases in the vertical direction. As a consequence, the cohomology level direct limit $SH^*_{vert}(M)$ is in general not isomorphic to the symplectic cohomology $SH^*(M)$. The same notational convention will be used later on for equivariant Floer cohomologies defined using a small time-dependent perturbation of $H_M$.

Let $M$ be a Liouville manifold. Recall that the action functional $\mathcal{A}_{H_t} : \mathcal{L}M \to \mathbb{R}$ of a time-dependent perturbation $H_t : S^1 \times M \to \mathbb{R}$ of some autonomous Hamiltonian $H : M \to \mathbb{R}$ is defined to be

$$\mathcal{A}_{H_t}(x) = -\int_{S^1} x^*\theta_M + \int_{S^1} H_t(x(t))dt. \tag{9.35}$$

Recall that the period spectrum $\mathcal{P}_M$ is a strictly ordered set with elements $0 = \eta_0 < \eta_1 < \eta_2 < \cdots$, where $\eta_1$ is the minimum period of a Reeb orbit on the contact boundary $\partial\overline{M}$. We set $\lambda_j = \frac{\eta_j+\eta_{j+1}}{2}$, so in particular $\lambda_j \notin \mathcal{P}_M$ for all $j$, and define the real numbers

$$a_{\lambda_j} := -\frac{\lambda_j^2}{2} - \lambda_j, j \geqslant 0. \tag{9.36}$$

Consider a Hamiltonian $H_{\lambda,t} \in \mathcal{H}_\lambda(M)$ so that $\lambda \notin \mathcal{P}_M$, let $\mathcal{O}_{M,\lambda}$ be the set of 1-periodic orbits of $X_{H_{\lambda,t}}$, there is an action filtration $F^\bullet$ on the Floer complex $CF^*(\lambda)$ of $H_{\lambda,t}$ given by

$$F^j CF^*(\lambda) := \bigoplus_{x\in\mathcal{O}_{M,\lambda}, \mathcal{A}_{H_{\lambda,t}}(x)\geqslant a_{\lambda_j}} |o_x|_{\mathbb{K}} \tag{9.37}$$

In order to analyze the compatibility between the $S^1$-complex structure maps $\{\delta_i\}_{i\geqslant 0}$ and the filtration $F^\bullet$ on $CF^*(\lambda)$, we study a specific autonomous Hamiltonian $\widetilde{H}_\lambda : M \to \mathbb{R}$, which is a $C^2$-small Morse function in the interior of $\overline{M}$, takes the form $\frac{(r-1)^2}{2}$ on the collar neighborhood $[1, \lambda+1] \times \partial\overline{M}$, and finally, $\widetilde{H}_\lambda(x) = \lambda(r-1) - \frac{\lambda^2}{2}$ is linear on the cylindrical end $[\lambda+1, \infty) \times \partial\overline{M}$. Following Section 3.2.1 of [117], we define a carefully-chosen small time-dependent perturbation $\widetilde{H}_{\lambda,t}$ of $\widetilde{H}_\lambda$. For any 1-periodic orbit $x \in \widetilde{\mathcal{O}}$ of $X_{\widetilde{H}_\lambda}$, fix an isolating neighborhood $U_x \subset M$. If $x$ corresponds to a Reeb orbit of multiplicity $k \in \mathbb{N}$, one considers a Morse function $f : S^1 \to \left[-1, -\frac{1}{2}\right]$ that has a unique minimum $f(0) = -1$ and a unique maximum $f(t_0) = -1/2$ for some small enough $t_0 \in S^1$. Define $h_x : \overline{U}_x \to [-1, 0]$ by

$$h_x(t, x(s)) = f(ks - kt) \tag{9.38}$$

on the image of $x$ and extend it smoothly to $U_x$ so that $h_x = 0$ on $\partial \overline{U}_x$. The time-dependent perturbation of $\widetilde{H}_\lambda$ is defined to be

$$\widetilde{H}_{\lambda,t} = \widetilde{H}_\lambda + \varepsilon' \sum_{x \in \widetilde{\mathcal{O}}} h_x(t), \tag{9.39}$$

where $h_x(t) = f_x \circ \phi^{-t}_{\widetilde{H}_\lambda}$, with $\phi^{-t}_{\widetilde{H}_\lambda}$ being the time $-t$ flow of $X_{\widetilde{H}_\lambda}$, and $\varepsilon' > 0$ is a small positive number.

With our choice of $\widetilde{H}_{\lambda,t}$, an energy estimate for Floer trajectories in $\mathcal{M}_i(y^+; y^-)$ with $|o_{y^+}|_{\mathbb{K}} \in F^j CF^*(\lambda)$ implies the following:

**Lemma 9.2.1** (Lemma 3.2.4 of [117])**.** *For any fixed $\lambda \notin \mathcal{P}_M$, there is an $\varepsilon' > 0$ depending on $\lambda$ such that*

$$\delta_i \left(F^j CF^*(\lambda)\right) \subset F^j CF^*(\lambda) \tag{9.40}$$

*for all $i, j \geqslant 0$.*

*Proof of Theorem 1.3.1.* We first show that (9.34) is a short exact sequence of $S^1$-complexes. Basically, (9.34) follows from the fact that the generators of $CF^*_{vert}(M, \lambda)$ consist of the following three kinds:

- Critical points of $H_F$. These generators have small negative action, if we perturb $H_F$ so that it is a $C^2$-small Morse function in the interior of $\overline{F}$.

- Non-constant 1-periodic orbits of $H_F$. Writing $H_F = h_F(r)$ on the cylindrical end $[1, \infty) \times \partial\overline{F}$, such an orbit $x$ has action

$$\mathcal{A}_{H_F}(x) = h_F(r_x) - r_x h_F'(r_x) < 0, \tag{9.41}$$

where $r_x \in [1, \infty)$ is the radial coordinate of $x$.

- Constant orbits near the critical points of $\pi^* H_{\mathbb{C}}$, with Conley-Zehnder index $-n$. If we choose the $C^2$-small Morse function appearing in the first item to be sufficiently small, it can be achieved that

$$\mathcal{A}_{H_M}(x) = \varepsilon|\pi(x) - c|^2 + H_F(x) > 0. \tag{9.42}$$

By choosing the $t$-dependent perturbation of the Hamiltonian $H_F$ (and thus $H_M$) carefully, Lemma 9.2.1 applies and shows that the operations $\{\delta_i\}_{i \geqslant 0}$ preserve the action filtration on $CF^*_{vert}(M, \lambda)$. This implies that the generators of $CF^*_{vert}(M, \lambda)$ with positive actions form the (trivial) $S^1$-subcomplex $\mathbb{K}^{\mathrm{Crit}(\pi)}[-n]$. On the other hand, the same energy estimate as in the proof of Lemma 7.2 of [103] shows that for any solution $u : Z \to M$ of the Floer equation $\left(du - X_{H_Z^{Lef}} \otimes \nu_Z\right)^{0,1} = 0$ with asymptotics $x^-, x^+ \in \mathcal{O}_{F,\lambda}$, its image necessarily lies in the fiber $F$. Here, $\nu_Z \in \Omega^1(Z)$, $H_Z^{Lef} \in \mathcal{H}^{Lef}(M)$, $J_Z^{Lef} \in \mathcal{J}^{Lef}(M)$ involved in the Floer equation come from the Floer data chosen for the moduli spaces $\overline{\mathcal{M}}_i$ of $i$-point angle decorated cylinders. However, the choices of Floer data here differ slightly from what we have seen in Definition 7.1.2 in the sense that they need to be compatible with the Lefschetz fibration structure. More precisely, $\mathcal{H}^{Lef}(M)$ is the space of Hamiltonians which are small $t$-dependent perturbations of the autonomous Hamiltonians taking the specific form (9.33), and $\mathcal{J}^{Lef}(M)$ is the space of compatible almost complex structures which are of contact type when restricted to the fibers $F$, and $\pi : M \to \mathbb{C}$ must be $(J, j)$-holomorphic.

The fact that $u(Z) \subset F$ implies that the quotient complex $CF^*_{vert}(M,\lambda)/\mathbb{K}^{\mathrm{Crit}(\pi)}[-n]$ can be identified with $CF^*(F,\lambda)$ as an $S^1$-complex. This proves that (9.34) is a short exact sequence of $S^1$-complexes, and it follows from Proposition 2.6.2 that we have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{K}((u))/u\mathbb{K}[[u]]^{\mathrm{Crit}(\pi)}[-n] \arrow[r] \arrow[d, "\mathbf{B}"] & HF^*_{S^1,vert}(M,\lambda) \arrow[r] \arrow[d, "\mathbf{B}"] & HF^*_{S^1}(F,\lambda) \arrow[r] \arrow[d, "\mathbf{B}"] & \cdots \\
\cdots \arrow[r] & \mathbb{K}^{\mathrm{Crit}(\pi)}[-n] \arrow[r] & HF^{*-1}_{vert}(M,\lambda) \arrow[r] & HF^{*-1}(F,\lambda) \arrow[r] & \cdots
\end{tikzcd} \tag{9.43}$$

whose rows are long exact sequences of $S^1$-equivariant and ordinary Floer cohomology groups, which are related through the marking map $\mathbf{B}$.

Now suppose that $F$ admits a cyclic dilation, so that for some $\lambda \gg 0$ and $\lambda \notin \mathcal{P}_F$, there is a class $\tilde{b}_F \in HF^1_{S^1}(F,\lambda)$, whose image under the composition of the marking map $\mathbf{B}$ and the continuation map defines an invertible element $h_F \in SH^0(F)^\times$. By our assumption that $n > 1$ is odd, the map $HF^1_{S^1,vert}(M,\lambda) \to HF^1_{S^1}(F,\lambda)$ in (9.43) is surjective, and we have the isomorphism

$$HF^0_{vert}(M,\lambda) \cong HF^0(F,\lambda), \tag{9.44}$$

which induces an isomorphism $SH^0_{vert}(M) \cong SH^0(F)$ of $\mathbb{K}$-algebras after passing to the direct limit. By the surjectivity of $HF^1_{S^1,vert}(M,\lambda) \to HF^1_{S^1}(F,\lambda)$ and the commutative diagram (9.43), $\tilde{b}_F$ lifts to a class $\tilde{b}_M \in HF^1_{S^1,vert}(M,\lambda)$, whose image under $\mathbf{B}$, followed by the continuation map, is the lift of $h_F$ in $SH^0_{vert}(M)$. In view of the isomorphism $SH^0_{vert}(M) \cong SH^0(F)$, this defines an invertible element of $SH^0_{vert}(M)$. Finally, there is an equivariant continuation map (whose construction is similar to that of (7.33))

$$SC^*_{vert}(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]] \to SC^*(M) \otimes_{\mathbb{K}} \mathbb{K}((u))/u\mathbb{K}[[u]], \tag{9.45}$$

under which the class $\tilde{b}_M$ goes to a cyclic dilation in $SH^1_{S^1}(M)$. $\square$

## 9.3 Varieties of log general type

We prove Theorem 1.4.2 in this section. Our argument is based on the work of McLean [79] on the symplectic invariance of the log Kodaira dimension, and the techniques in [15, 56], which allow us to produce $J$-holomorphic curves starting from Floer trajectories. For completeness, we shall start by recalling some of the important notions and results from [79].

Let $(\overline{M}, \theta_M)$ be any Liouville domain and let $J$ be an almost complex structure on $\overline{M}$ which is compatible with the symplectic form $d\theta_M$. It is *convex* if there is some function $\phi : \overline{M} \to \mathbb{R}$ such that

- $\partial\overline{M}$ is a regular level set of $\phi$ and $\phi$ attains its maximum on $\partial\overline{M}$;
- $\theta_M \circ J = d\phi$ near $\partial\overline{M}$.

Every Liouville domain $\overline{M}$ has a convex almost complex structure since one can take $\phi = r$ to be the radial coordinate function in a collar neighborhood of $\partial\overline{M}$, and then extend it smoothly to the interior. The following notion plays a pivotal role in McLean's theory.

**Definition 9.3.1** ([79], Definition 2.2)**.** *Let $k \in \mathbb{Z}_{>0}$, and let $\mu \in \mathbb{R}_{>0}$. A Liouville domain $\overline{M}$ is $(k,\mu)$-uniruled if for every convex almost complex structure $J$ and every point $p \in M^{in}$ so that $J$ is integrable in a neighborhood of $p$, there is a proper $J$-holomorphic map $u : S \to M^{in}$ whose image passes through $p$, where $S$ is a genus 0 Riemann surface with* $\dim H_1(S;\mathbb{Q}) \leqslant k-1$, *and the energy of $u$ is at most $\mu$.*

It follows from Theorem 2.3 of [79] that $(k,\mu)$-uniruledness is a symplectic invariant of the completion after forgetting about the energy bound $\mu$.

We now restrict ourselves to the special case when $M$ is an $n$-dimensional smooth affine variety. We say that $M$ is *algebraically $k$-uniruled* if there is a polynomial map $S \to M$ passing through every generic point $p \in M$, where $S$ is $\mathbb{CP}^1$ with at most $k$ points removed. This notion of uniruledness is related to Definition 9.3.1 as follows.

**Theorem 9.3.1** ([79], Theorem 2.5)**.** *Let $M$ be a smooth affine variety. If the associated Liouville domain $\overline{M}$ is $(k, \mu)$-uniruled for some $\mu$, then $M$ is algebraically $k$-uniruled.*

$k$-uniruledness of an affine variety is closely related to its log Kodaira dimension. In particular, we have the following:

**Lemma 9.3.1** ([79], Lemma 7.1)**.** *Let $M$ be a smooth affine variety which is algebraically $k$-uniruled. If $k = 1$, then $\kappa(M) = -\infty$, and if $k = 2$, then $\kappa(M) \leqslant n - 1$.*

*Proof of Theorem 1.4.2.* Suppose that $M$ is an $n$-dimensional smooth affine variety so that $SH^0(M)^\times$ is not isomorphic to $\mathbb{K}^\times$, or equivalently, there is an $h \in SH^0(M)^\times$ which is not a multiple of the identity. Since $hh^{-1} = 1$ holds in $SH^0(M)$, there must be some generator $\eta$ of $SC^0_+(M)$ so that $\alpha \cdot e + \eta$ is invertible when passing to cohomology, where $\alpha \in \mathbb{K}$ and $SC^0_+(M) \subset SC^0(M)$ is the submodule generated by non-constant Hamiltonian orbits. Let $L \subset M$ be an exact Lagrangian $K(\pi, 1)$, consider the Viterbo restriction map $SH^*(M) \to H_{n-*}(\mathcal{L}L; \mathbb{K})$, under which $\alpha \cdot e + \eta$ becomes a non-trivial central unit

$$\alpha \cdot \iota_*[L] + h_L \tag{9.46}$$

in the fundamental group algebra $\mathbb{K}[\pi_1(L)]$, where $\iota : L \to \mathcal{L}L$ is the inclusion of constant loops and $h_L$ is some central element. By Theorems 6.1.2 and 6.1.3 of [30], any non-trivial central unit of $\mathbb{K}[\pi_1(L)]$ must satisfy $\alpha = 0$. It follows that the pairing

$$SC^0_+(M) \otimes SC^0_+(M) \xrightarrow{\smile} SC^0(M) \xrightarrow{pr} C^0(M;\mathbb{K}) \cong \mathbb{K} \tag{9.47}$$

defined by composing the pair-of-pants product with the natural projection to the subcomplex $C^0(M;\mathbb{K}) \subset SC^0(M)$ does not vanish, so there must be some $y_0^+, y_1^+ \in \mathcal{O}_M$, such that $y_1^+ \smile y_0^+ = \alpha' \cdot e$, where $\alpha' \in \mathbb{K}^\times$ is some non-zero scalar. Without loss of generality, we may assume that $y_1^+ \smile y_0^+ = e$ for convenience.

This implies the existence of a map $u : S \to M$, with $S$ being a 3-punctured sphere, which satisfies the Floer equation $(du - X_{H_S} \otimes \nu_S)^{0,1} = 0$, and is asymptotic to the non-constant periodic orbits $y_0^+, y_1^+ \in \mathcal{O}_M$ at two positive cylindrical ends, and converges to the minimum $y^-$ of some $C^2$-small Morse function defined on $M^{in}$. Here $H_S : S \to \mathcal{H}(M)$ is a domain-dependent Hamiltonian-function so that its restriction to $M^{in}$ is a (domain-independent) $C^2$-small Morse function, and $\nu_S \in \Omega^1(S)$ is a closed 1-form, they are fixed as part of our Floer data defining the pair-of-pants product, and the $(0,1)$-part in the Floer equation is taken with respect to some domain-dependent almost complex structure $J_S : S \to \mathcal{J}(M)$. Starting from this, one can apply a limiting argument of [15, 56] to produce a $J$-holomorphic cylinder

$$\bar{u}_\infty : Z \to M^{in}_{1-\varepsilon} \tag{9.48}$$

with finite energy which passes through $y^-$ for any convex almost complex structure $J$ on the slightly shrinked Liouville domain

$$\overline{M}_{1-\varepsilon} := \overline{M} \backslash (1-\varepsilon, 0] \times \partial \overline{M} \tag{9.49}$$

containing $y^-$ in its interior, and whose completion is still deformation equivalent to $M$, where $\varepsilon > 0$ is a small constant.

To do this, we work with linear Hamiltonians instead, and introduce a particular 1-parameter family of domain-dependent Hamiltonians $H_{\lambda,S,\theta} : S \to \mathcal{H}_\lambda(M)$ which depend on a small parameter $\theta > 0$, where as before $\lambda \notin \mathcal{P}_M$ and $\lambda \gg 0$. Specifically, for each point $z \in S$, there is a Hamiltonian

$$H_{\lambda,z,\theta} = H_{\lambda,\theta} + F_{\lambda,z} \in \mathcal{H}_\lambda(M), \tag{9.50}$$

where $F_{\lambda,z} : S \times M \to \mathbb{R}$ is independent of $s$ when restricted to the cylindrical ends, and since it is supported near non-constant orbits of $X_{H_{\lambda,\theta}}$, we can choose $\varepsilon > 0$ so that $F_{\lambda,z}$ vanishes on $\overline{M}_{1-\varepsilon}$. Note that $\varepsilon > 0$ can be taken to be arbitrarily small. Set

$$H_{\lambda,\theta} = \begin{cases} -C_\lambda + \theta f & \text{on } M^{in} \\ h_{\lambda,\theta}(r) & \text{on } \overline{M}_{1+2\varepsilon}\backslash M^{in} \\ \lambda(r-1-\varepsilon) & \text{on } M\backslash\overline{M}_{1+2\varepsilon} \end{cases} \tag{9.51}$$

where $C_\lambda > 0$ is a small scalar which satisfies $\lim_{\lambda\to\infty} C_\lambda = 0$, $f$ is a $C^2$-small Morse function when restricted to $M^{in}_{1-\varepsilon}$, satisfying $-1 \leqslant f \leqslant 0$ on $\overline{M}_{1-\varepsilon}$, has a relative minimum at $y^- \in M^{in}_{1-\varepsilon}$, and equals $r - 1 + \varepsilon$ on $[1-2\varepsilon, 1] \times \partial\overline{M}$. $h_{\lambda,\theta}(r)$ is an arbitrary convex function on $[1, 1+2\varepsilon] \times \partial\overline{M}$ which depends only on $r$, and whose slope varies from $\theta$ to $\lambda$ as $r$ goes from 1 to $1+2\varepsilon$, such that $h_\lambda(r) := \lim_{\theta\to 0} h_{\lambda,\theta}(r)$ is a smooth function.

With our particular choice of the domain-dependent Hamiltonian $H_{\lambda,S,\theta}$ as above, we get a Floer trajectory $u_{\lambda,\theta} : S \to M$ which is asymptotic to a Morse critical point $y^-_\theta$ at its negative cylindrical end, and to $y^+_{0,\theta}, y^+_{1,\theta} \in \mathcal{O}_{M,\lambda}$ at two positive cylindrical ends. It follows from our definition of $H_{\lambda,\theta}$ that the non-constant orbits $y^+_{0,\theta}, y^+_{1,\theta}$ necessarily lie in the collar $[1, 1+2\varepsilon] \times \partial\overline{M}$. To achieve the non-degeneracies of the orbits $y^+_{0,\theta}$ and $y^+_{1,\theta}$, the perturbation $F_{\lambda,z}$ can be taken to be supported near $y^+_{0,\theta}$ and $y^+_{1,\theta}$, so we may assume (by possibly rescaling $\varepsilon$) that $H_{\lambda,z,\theta} = H_{\lambda,\theta}$ is domain-independent in the shrinked Liouville domain $\overline{M}_{1-\varepsilon}$. Applying the maximum principle from Section 7d of [7] to the map $u_{\lambda,\theta}$ shows that $u_{\lambda,\theta}(S) \subset \overline{M}_{1+2\varepsilon}$. To achieve transversality of the moduli space $\mathcal{P}\left(y_{0,\theta}^+, y_{1,\theta}^+; y_\theta^-\right)$ where the trajectory $u_{\lambda,\theta}$ lies in, one can start from any convex almost complex structure $J$ on $\overline{M}$ and perturb it slightly outside of $\overline{M}_{1-\varepsilon}$ to get a domain-dependent almost complex structure $J_{\lambda,S,\theta} : S \to \mathcal{J}(M)$. Note that we have arranged so that both of $H_{\lambda,S,\theta}$ and $J_{\lambda,S,\theta}$ are domain-independent on $\overline{M}_{1-\varepsilon}$, and we denote the restriction of $J_{\lambda,S,\theta}$ on $\overline{M}_{1-\varepsilon}$ as $J_{\lambda,\theta}$.

We want to pass to the limit $\theta \to 0$. Notice that when restricted to the Liouville domain $\overline{M}_{1-\varepsilon}$, we have $\lim_{\theta\to 0} H_{\lambda,\theta} = -C_\lambda$, and $\lim_{\theta\to 0} J_{\lambda,\theta} = J$, for some fixed convex almost complex structure $J \in \mathcal{J}(M)$ which doesn't need to depend on $\lambda$. By the same argument as in the proof of Proposition 5.11 of [56], one can find a sequence $\{\theta_n\}$ which limits to 0 so that the corresponding Floer trajectories satisfy $u_{\lambda,\theta_n} \to u_\lambda$ in $C^\infty_{\mathrm{loc}}(S, M)$, and the energy of the limiting trajectory

$$E\left(u_\lambda\right) := \frac{1}{2}\int_S \left|\left|du_\lambda - X_{H_{\lambda,z,0}} \otimes dt\right|\right|^2_{J_{\lambda,z,0}} \leqslant \mu_M \tag{9.52}$$

is bounded above by some constant $\mu_M > 0$, which is independent of $\lambda$. Denote by $\phi$ a biholomorphic map which identifies $S$ with $\mathbb{CP}^1\backslash\{0,1,\infty\}$ so that the negative puncture $\zeta_{out}$ is mapped to the origin. The composition $\tilde{u}_\lambda = u_\lambda \circ \phi : \mathbb{CP}^1\backslash\{0,1,\infty\} \to M$ is a map whose limit at the origin is $y^- := \lim_{\theta\to 0} y_\theta^-$ and which goes outside of $M^{in}$ when approaching the other two punctures. Note that by our choice of $H_{\lambda,\theta}$, the minimum $y_\theta^-$ of the $C^2$-small Morse function $f$ is independent of $\theta > 0$, so we actually have $y^- \in M^{in}_{1-2\varepsilon}$. On the other hand, it also follows from our choice of $H_{\lambda,\theta}$ that $y_0^+ := \lim_{\lambda\to\infty} y_{0,\theta}^+$ and $y_1^+ := \lim_{\lambda\to\infty} y_{1,\theta}^+$ fall outside of $M^{in}$.

Pick any $R_\lambda \in (1-2\varepsilon, 1-\varepsilon)$, and consider the inverse image $\tilde{u}_\lambda^{-1}(\overline{M}_{R_\lambda})$. As $\lambda \to \infty$, $\lim_{\lambda\to\infty} R_\lambda = 1-\varepsilon$. Since $y^- \in M^{in}_{1-2\varepsilon}$ and $y_0^+, y_1^+ \notin M^{in}$, $\tilde{u}_\lambda^{-1}(M^{in}_{R_\lambda}) \subset \mathbb{CP}^1\backslash\{0,1,\infty\}$ is an open punctured cylinder for some $\varepsilon > 0$ which can be taken sufficiently small. We will denote it by $Z_\lambda^* \subset Z$. Since $H_\lambda \equiv C_\lambda$ in $M^{in}$, it follows that the map

$\tilde{u}_\lambda : Z_\lambda^* \to M_{R_\lambda}^{in}$ is $J$-holomorphic. Moreover, we have

$$\int_{\partial \overline{Z}_\lambda^*} \tilde{u}_\lambda^* \theta_M \leqslant E(u_\lambda) \leqslant \mu_M, \tag{9.53}$$

where $\theta_M$ is the Liouville form. In particular, the removable singularity theorem for pesudoholomorphic maps applies, which shows that $\tilde{u}_\lambda$ extends to a $J$-holomorphic map $\bar{u}_\lambda : Z_\lambda \to M_{R_\lambda}^{in}$. Letting $\lambda \to \infty$, we get a $J$-holomorphic map $\bar{u}_\infty : Z \to M_{1-\varepsilon}^{in}$ whose image passes through $y^-$. The uniruledness of the Liouville domain $\overline{M}_{1-\varepsilon}$ follows by noticing that $y^-$ can be taken to be any generic point in $M_{1-\varepsilon}^{in}$. Alternatively, one can argue as follows.

A slight variation of the construction of the moduli space $\mathcal{P}(y_0^+, y_1^+; y_-)$ enables us to define

$$\mathcal{P}(y_0^+, y_1^+; \overline{M}), \tag{9.54}$$

which parametrizes maps $u : S \to M$ satisfying Floer's equation, but are now asymptotic to $y_0^+, y_1^+ \in \mathcal{O}_{M,\lambda}$ at two positive ends, and converges to the relative fundamental cycle in $C_{2n}(\overline{M}, \partial \overline{M})$ at the negative end. The Gromov bordification of $\mathcal{P}(y_0^+, y_1^+; \overline{M})$ carries an evaluation map

$$\overline{ev} : \overline{\mathcal{P}}\left(y_0^+, y_1^+; \overline{M}\right) \to \overline{M} \tag{9.55}$$

defined to be the asymptote at the negative puncture for every $u \in \mathcal{P}\left(y_0^+, y_1^+; \overline{M}\right)$, and the coefficient before the identity $e \in CF^0(2\lambda)$ under the pair-of-pants product $y_1^+ \smile y_0^+$ is defined by pushing forward the fundamental chain $\left[\overline{\mathcal{P}}\left(y_0^+, y_1^+; \overline{M}\right)\right]$ via $\overline{ev}$. Our assumption that $h$ defines a non-trivial unit in $SH^0(M)$ implies that for appropriate choices of Floer data, there is an identification between $\overline{\mathcal{P}}\left(y_0^+, y_1^+; \overline{M}\right)$ and $\overline{M}$ relative to the boundaries. Applying the same argument as above to every element $u$ of $\mathcal{P}\left(y_0^+, y_1^+; \overline{M}\right)$ proves that the Liouville domain $\overline{M}_{1-\varepsilon}$ is $(2, \mu_M)$-uniruled in the sense of Definition 9.3.1. It follows from Theorem 9.3.1 that $M$ is algebraically 2-uniruled.

By Lemma 9.3.1, $M$ cannot be of log general type. In other words, for any smooth affine variety $M$ of log general type which contains an exact Lagrangian $K(\pi, 1)$, we necessarily have $SH^0(M)^\times \cong \mathbb{K}^\times$. Appealing to Corollary 8.1.1 completes our proof. $\square$

Note that our theorem provides an alternative way to understand Corollary 8.1.2. One can also try to prove a statement of similar flavour as Theorem 1.4.2 by making use of the *logarithmic PSS map* introduced by Ganatra-Pomerleano in [48, 49]. Under the assumption that $M = X \backslash D$, where $(X, D)$ is a *multiplicatively topological pair* in the sense of [49], $SH^0(M)$ is isomorphic to the *logarithmic cohomology* $H^*_{log}(X, D)$ as a $\mathbb{K}$-algebra, while $H^0_{log}(X, D)$ does not contain any non-trivial unit.

## 9.4 A conjectural picture

Although the results obtained in this chapter are far from providing a complete classification of Liouville manifolds admitting cyclic dilations, in view of our discussions in Section 1.4, it seems to be reasonable to expect the following (note that we consider here only the case when $\mathrm{char}(\mathbb{K}) = 0$):

**Conjecture 9.4.1.** *Let $M$ be an $n$-dimensional smooth affine variety.*

- *If $\kappa(M) = -\infty$, then $M$ admits a cyclic dilation with $h = 1$.*
- *If $\kappa(M) = 0$, then $M$ admits a cyclic dilation if and only if it admits a quasi-dilation with $h \neq 1$.*
- *If $\kappa(M) = n$, then $M$ does not admit a cyclic dilation.*

Note that in order for our conjecture to make sense, we need to regard manifolds with $SH^*(M) = 0$ as manifolds which carry cyclic dilations. As is conjectured by McLean, this is precisely the case when $M$ is *ruled* by affine lines.

The expectation that cyclic dilations should exist for all affine varieties with $\kappa(M) = -\infty$ is probably too optimistic, it seems to be more reasonable to state the conjecture for all the Milnor fibers with $\kappa(M) = -\infty$. However, there are affine varieties with $\kappa(M) = -\infty$ which are not Milnor fibers, but which do admit cyclic dilations. For example, we ave the affine hypersurface $M_{1,1,0} \subset \mathbb{C}^4$ mentioned in Section 6.4. We know that $M_{1,1,0}$ admits a cyclic dilation because it carries a Lefschetz fibration $\pi : M_{1,1,0} \to \mathbb{C}$ with the smooth fiber being symplectomorphic to a 4-dimensional $D_4$ Milnor fiber, so combining our argument in Section 3.5 with the Lefschetz fibration method due to Seidel-Solomon [103] shows that $M$ admits a quasi-dilation. This example is also interesting in the sense that the existence of an exact Calabi-Yau structure on $\mathcal{W}(M)$ does not follow from Van den Bergh's Theorem 9.1.1. In fact, it follows from (6.176) that the wrapped Fukaya $A_\infty$-algebra $\mathcal{W}_{1,1,0}$ of $M_{1,1,0}$ is quasi-isomorphic to a (formal) dg algebra with generators in positive degrees, so Theorem 9.1.1 is not applicable.

According to Theorem 1.5.1, one may expect that every Liouville manifold $M$ admitting a cyclic dilation has an associated number $N$, the maximal number of disjoint Lagrangian spheres (although at present I don't know how to prove this without assuming property ($\widetilde{\mathrm{H}}$)). This should be viewed as a kind of "capacity" measuring the size and complexity of $M$. It is therefore natural to expect that the bound $N$ is related to certain kind of symplectic capacity. Here, the most relevant to us is the *Gutt-Hutchings capacities* [54], which is a sequence of symplectic capacities

$\left\{c_k^{GH}(M)\right\}_{k\geqslant 1}$ defined to be

$$c_k^{GH}(M) := \inf\left\{a|\delta_{eq}(x) = u^{-k+1}e \text{ for some } x \in F_{\leqslant a}SC_{S^1}^{-2k+1}(M)\right\}, \tag{9.56}$$

where $F_\bullet$ is the action filtration. It follows from our discussions in Section 7.2 that $M$ admits a cyclic dilation with $h = 1$ if and only if $c_1^{GH}(M) < \infty$, and admits a higher dilation if and only if $c_k^{GH}(M) < \infty$ for any $k \geqslant 1$.[1] Thus the first item of Conjecture 9.4.1 follows from the following:

**Conjecture 9.4.2.** *Let $M$ be a smooth affine variety with $\kappa(M) = -\infty$, then $c_1^{GH}(M) < \infty$.*

It seems likely that there is no exact Lagrangian tori in smooth affine varieties with $\kappa(M) = -\infty$. In view of Corollary 8.1.1, this provides evidences for the more precise expectation that the marking map $\mathbf{B} : SH_{S^1}^1(M) \to SH^0(M)$ should actually hit the identity.

If all the smooth affine varieties which admit cyclic dilations are those with $h = 1$, and those admitting quasi-dilations, then one can normalize the definition of the endomorphism $\Phi_{\tilde{L},\tilde{L}}$ on $HF^*(L, L)$ for a $\tilde{b}$-equivariant Lagrangian sphere $\widetilde{L} = (L, \tilde{\gamma}_L)$ so that it acts as the identity on $HF^n(L, L)$. However, it remains unclear whether a smooth affine variety with $0 < \kappa(M) < n$ could admit a cyclic dilation. Note that contractible affine surfaces with log Kodaira dimension 1 are classified, see for example [108]. For these surfaces, one can show (using techniques of [80]) that there is a non-trivial invertible element in $SH^0(M)$. However, they do not admit cyclic dilations as each of them contains $\mathbb{D}^* \times \overline{F}$ as a Liouville subdomain, where $\mathbb{D}^*$ is the punctured disc, and $F$ is a once punctured surface with genus $g \geqslant 1$.

[1] However, this observation does not lead to a proof that $M$ cannot contain infinitely many disjoint Lagrangian spheres, as is explained in Remark 1.27 of [54].

Since there should be an exact Lagrangian torus in every smooth log Calabi-Yau variety, one expects that $h \neq 1$ in view of Corollary 8.1.1. This can be rigorously proved for log Calabi-Yau surfaces, in which case it follows essentially from Proposition 1.3 of [53] that every log Calabi-Yau surface with maximal boundary is obtained as $D^*T^2$ with some 2-handles attached along its boundary.

By Theorem J of [118], if a smooth affine variety $M$ admits a dilation, then $\overline{M}$ is $(1, \mu)$-uniruled for some $\mu > 0$ in the sense of Definition 9.3.1. In particular, $\kappa(M) = -\infty$. It seems the same argument as in Section 5 of [118] can be applied to prove the uniruledness of $M$ by affine lines when it admits a cyclic dilation with $h = 1$. Combining with our Theorem 1.4.2, this proves the non-existence of cyclic dilations for affine varieties with $\kappa(M) = n$.

# Appendix A

# Computing the signs

## A.1 Orientations of Morse flow trees

This section reviews the works [58, 59] of Karlsson, which allow us to define the Chekanov-Eliashberg algebra $CE^*(\Lambda)$ over $\mathbb{Z}$ for a Legendrian submanifold $\Lambda \subset J^1(S)$. Given a rigid Morse flow tree $\Gamma$, the definition of the sign $\varepsilon(\Gamma)$ involves four independent signs, namely the sign $\nu_{triv}(\Gamma)$ which depends on the choice of *Spin* structures on $\Lambda$; the sign $\nu_{int}(\Gamma)$ which records the intersection orientation of the flow-outs of the sub flow trees of $\Gamma$; the sign $\nu_{end}(\Gamma)$ which encodes the information of $e$-vertices; and the sign $\nu_{stab}(\Gamma)$ which comes from capping orientations of pseudo-holomorphic discs associated to Morse flow trees. Since we are only interested in the case when $\Lambda \subset \mathbb{R}^5$ is a disjoint union of Legendrian 2-spheres, the first sign $\nu_{triv}(\Gamma)$ is irrelevant for us, therefore we only recall here the definitions of $\nu_{int}(\Gamma)$, $\nu_{end}(\Gamma)$ and $\nu_{stab}(\Gamma)$.

As in Chapter 5, let $\Lambda \subset J^1(S)$ be a Legendrian surface which is *Spin*, where $S$ is an orientable surface equipped with a Riemannian metric $g$. Let $S_i$ and $S_j$ be two sheets of $\Lambda$ over a small open subset $U \subset S$, and assume that $S_i$ lies above $S_j$, namely $z(S_i) > z(S_j)$. Denote by $f_i$ and $f_j$ the local defining functions of $S_i$ and $S_j$ respectively, and set $f_{ij} := f_i - f_j$. We refer to the fundamental paper [32] for background materials concerning Morse flow trees.

**Definition A.1.1.** *Let $\ell$ be a flow line of the local difference function $f_{ij}$, and let $K \subset S$ be a subset with $K \cap \ell \neq \varnothing$. The flow-out of $K$ along $\ell$ is the union of all maximal flow lines of $-\nabla f_{ij}$ that intersect $K$.*

Let $c$ be a puncture of the rigid Morse flow tree $\Gamma$ with $f_i > f_j$, denote by $\mathcal{U}(c)$ and $\mathcal{S}(c)$ respectively the unstable and stable manifold of $-\nabla f_{ij}$. Given a sub flow tree $\Gamma' \subset \Gamma$, denote by $s$ its special puncture. There is an edge $\ell \subset \Gamma'$ ending at $s$, with the other end point given by some true vertex $t$ of $\Gamma$. Denote by $\mathcal{P}^+, \mathcal{P}^-, \mathcal{E}, \mathcal{P}^2$ and $\mathcal{Y}_0$ the set of 1-valent positive punctures, 1-valent negative punctures, $e$-vertices, negative 2-valent punctures, and $Y_0$-vertices respectively. The *flow-out of* $\Gamma'$ *at* $s$, denoted by $FO_s(\Gamma')$, is defined as follows.

- When $t \in \mathcal{P}^+$, $FO_s(\Gamma')$ is the flow-out of $t$ along $\ell$. In particular, we have an identification between the tangent spaces $T_sFO_s(\Gamma') \cong T_s\mathcal{U}(t)$.

- When $t \in \mathcal{P}^-$, $FO_s(\Gamma')$ is the flow-out of $t$ along $\ell$. In particular, $T_sFO_s(\Gamma') \cong T_s\mathcal{S}(t)$.

- When $t \in \mathcal{E}$, let $I_t$ be an open interval centred at $t$ which is transverse to $\ell$, $FO_s(\Gamma')$ is the flow out of $I_t$ along $\ell$.

- When $t \in \mathcal{Y}_0$, and $s$ is a special positive puncture of the sub flow tree $\Gamma'$, the definition of $FO_s(\Gamma')$ is given inductively. Define the *intersection manifold*

$$IM_t(\Gamma') := FO_t(\Gamma'_1) \cap FO_t(\Gamma'_2), \tag{A.1}$$

where $\Gamma'_1$ and $\Gamma'_2$ are the sub flow trees of $\Gamma$ with $t$ as their common special positive puncture. $FO_s(\Gamma')$ is defined to be the flow-out of $IM_t(\Gamma') \cap I_t$ along $\ell$;

- When $t \in \mathcal{P}^2$, the intersection manifold $IM_t(\Gamma') := \{t\}$ and $FO_s(\Gamma') = \ell$.

The cases when $t$ is a switch or a $Y_1$-vertex are not recalled here, as we will not encounter rigid Morse flow trees with such kind of internal vertices in the computation of $CE^*(\Lambda_{p,q,r})$ in Section A.2. To simplify the discussions, assume from now on that there is *no* switch or $Y_1$-vertex in $\Gamma$.

**Remark A.1.1.** *We should emphasize that the sub flow trees $\Gamma'_1$ and $\Gamma'_2$ are numbered so that the standard domain of $\Gamma'_1$ corresponds to the lower part in the standard domain of $\Gamma'$.*

To define the intersection orientation sign $\nu_{int}$, we start from some basic facts in linear algebra. Let $V_1, V_2 \subset \mathbb{R}^n$ be subspaces, and $V = V_1 \cap V_2$. There is a short exact sequence

$$0 \to V \xrightarrow{\delta} V_1 \oplus V_2 \xrightarrow{\eta} \mathbb{R}^n \to 0, \tag{A.2}$$

where

$$\delta(v) = (v, v), \eta(u, v) = v - u. \tag{A.3}$$

For fixed choices of orientations on $V_1$ and $V_2$, there is an orientation $o(V)$ on their intersections such that the orientation induced by $V_i$ on its quotient $V_i/V$, together with $o(V)$, coincides with the original orientation on $V_i$. Define $\nu \in \{0, 1\}$ to be the sign which satisfies

$$V \oplus (V_1/V) \oplus (V_2/V) \cong (-1)^\nu \cdot \mathbb{R}^n \tag{A.4}$$

as oriented vector spaces.

**Definition A.1.2.** *The intersection orientation* $o_{int}(V)$ *is given by*

$$o_{int}(V) := o_{int}(V_1, V_2) = (-1)^{\nu + \dim V_1 \cdot (1+\dim V)} o(V). \tag{A.5}$$

Back to our specific geometric set up, let $o_{cap}(\mathcal{U}(t))$ denote the initial choice of the orientation of $\mathcal{U}(t)$, given as a wedge product of an oriented basis of $T_s(\mathcal{U}(t))$. It induces an orientation $o_{cap}(T_s\mathcal{S}(t))$ on the stable manifold such that

$$o_{cap}(T_t\mathcal{U}(t)) \wedge o_{cap}(T_t\mathcal{S}(t)) \tag{A.6}$$

gives the original orientation on $T_tS$.

The orientations are defined inductively as follows:

- if $t \in \mathcal{P}^+$, then
$$o(FO_s(\Gamma')) := o_{cap}(T_s\mathcal{U}(t)); \tag{A.7}$$

- if $t \in \mathcal{P}^-$, then
$$o(FO_s(\Gamma')) := o_{cap}(T_s\mathcal{S}(t)); \tag{A.8}$$

- if $t \in \mathcal{E}$, then
$$o(FO_s(\Gamma')) = o(T_sS); \tag{A.9}$$

- if $t \in \mathcal{P}^2$, then
$$o(IM_t(\Gamma')) = o_{int}(o(FO_t(\Gamma'_1)), o_{cap}(\mathcal{U}(t))); \tag{A.10}$$

- if $t \in \mathcal{Y}_0$, $s$ is a special positive puncture of $\Gamma'$, then
$$o(IM_t(\Gamma')) = o_{int}(o(FO_t(\Gamma'_1)), o(FO_t(\Gamma'_2))). \tag{A.11}$$

In the above, the orientations of flow-outs when $t$ is a 2-valent puncture or a $Y_0$-vertex are defined in terms of intersection manifolds. To recover the orientation of the flow-out $FO_s(\Gamma')$, let $v_s$ be the tangent vector of $\ell$ at $s$, pointing in the direction against the defining gradient vector field, and define the orientation of the flow-out along $\ell$ as

$$o(FO_s(\Gamma')) = o(IM_t(\Gamma')) \wedge v_s, \tag{A.12}$$

where we have used parallel transport over the elementary regions to identify tangent spaces of $S$.

Assume that the positive puncture $a$ of $\Gamma$ is 1-valent. In this case, the sign $\nu_{int}$ can be defined as follows. Let $c$ denote the first vertex that we meet when going along $\Gamma$ from $a$, and let $\ell \subset \Gamma$ be an edge which starts at $c$, orient it so that it points toward $a$. Pick a point $s \in \ell$ which is contained in the same elementary region as $c$. Cutting at $s$ we obtain two sub flow trees

$$\Gamma = \Gamma'_1 \cup \Gamma'_2, \tag{A.13}$$

where $\Gamma'_1$ has $s$ as its positive special puncture, and since we have assumed that there is no switch in $\Gamma$, the unique true vertex of $\Gamma'_2$ is the positive puncture $a$.

If the standard domain $\Delta(\Gamma)$ has no slit, namely when $\Gamma'_1$ has a unique true vertex $t$, assume that the flow orientation of the edge $\ell$ connecting $s$ to $t$ is $\frac{\partial}{\partial x_1}$. The sign $\nu_{int}(\Gamma)$ is determined by the formula

$$o_{int}(o(FO_t(\Gamma'_1)), o(FO_t(\Gamma'_2))) = \nu_{int}(\Gamma) \cdot \frac{\partial}{\partial x_1}. \tag{A.14}$$

When the standard domain $\Delta(\Gamma)$ has at least one slit, which corresponds to the vertex $c \in \Gamma$. In the above, we have inductively defined the orientations $o(IM_c(\Gamma'_1))$ and $o(FO_s(\Gamma'_2))$. Using flat coordinates along the edge of $\Gamma$ connecting $c$ to $s$, we can identify $T_s IM_c(\Gamma'_1)$ with $T_c IM_c(\Gamma'_1)$, and $\nu_{int}(\Gamma)$ is defined via the formula

$$T_s S \wedge T_s S = \nu_{int}(\Gamma) \cdot o(IM_c(\Gamma'_1)) \wedge o(FO_s(\Gamma'_2)). \tag{A.15}$$

The sign $\nu_{end}$ records the information of $e$-vertices in $\Gamma$. Since its explicit form will not be needed for our purposes, we are not going to recall its definition, see [59] for details.

The sign $\nu_{stab}$ is not defined explicitly, but can be determined by referring to the capping exact sequence of Morse flow trees, see Section 6.2 of [58]. Basically, it is defined as a sum of contributions from punctures and internal vertices of different types. More precisely, since we have assumed the non-existence of switches and $Y_1$-vertices, we only need to consider the sets $\mathcal{P}^+, \mathcal{P}^-, \mathcal{P}^2$, and $\mathcal{Y}_0$. For any vertex $c$ in these sets, there is a well-defined sign $\sigma_{\mathcal{P}^+}(c), \sigma_{\mathcal{P}^-}(c), \sigma_{\mathcal{P}^2}(c)$ and $\sigma_{\mathcal{Y}_0}(c)$ respectively. With these notations, the sign $\nu_{stab}$ is given by

$$\nu_{stab}(\Gamma) = (-1)^{\sigma_{\mathcal{P}^+}(a)+\sum_{c\in\mathcal{P}^-}\sigma_{\mathcal{P}^-}(c)+\sum_{c\in\mathcal{P}^2}\sigma_{\mathcal{P}^2}(c)+\sum_{c\in\mathcal{Y}_0}\sigma_{\mathcal{Y}_0}(c)}. \tag{A.16}$$

The precise forms of the signs $\sigma_{\mathcal{P}^+}$, $\sigma_{\mathcal{P}^-}$, and $\sigma_{\mathcal{Y}_0}$ will be recalled below.

Fix a Maslov potential $\mu : \Lambda \to \mathbb{Z}$. For $c \in \Gamma$ a puncture or a special puncture, let $|\mu(c)| \in \mathbb{Z}/2$ denote the parity of its Maslov index. When $c$ is a critical point of some local difference function $f_{ij}$, we have a well-defined Morse index $ind(c)$.

If $\Gamma' \subset \Gamma$ is a sub flow tree with special puncture at $s \in \Gamma$, denote by $bm(\Gamma')$ the number of boundary minima in the standard domain $\Delta(\Gamma')$. If $bm(\Gamma') > 0$, let $ord(\Gamma')$ be the order of the boundary minimum of $\Delta(\Gamma')$ with the smallest $\kappa$-value. When $bm(\Gamma') = 0$, we set $ord(\Gamma') = 0$.

We introduce a subspace $\ker_s(\Gamma') \subset T_sFO_s(\Gamma')$, called the *true kernel* of $\Gamma'$, which is defined inductively as follows:

- when $t \in \mathcal{P}^+$, $\ker_s(\Gamma') = T_sFO_s(\Gamma')$;

- when $t \in \mathcal{P}^-$, $\ker_s(\Gamma') = T_s FO_s(\Gamma')$;
- when $t \in \mathcal{E}$, $\ker_s(\Gamma') = T_s FO_s(\Gamma')$;
- when $t \in \mathcal{P}^2$, $\ker_s(\Gamma') = 0$;
- when $t \in \mathcal{Y}_0$, $\ker_s(\Gamma') = \ker_t(\Gamma'_1) \cap \ker_t(\Gamma'_2)$.

In the above, $t$ is the true vertex of $\Gamma'$ which is connected to the special puncture $s$ by an edge $\ell \subset \Gamma'$, $\Gamma'_1, \Gamma'_2 \subset \Gamma$ are sub flow trees starting at the $Y_0$-vertex $t$, and we have used parallel transport to identify tangent spaces of flow-outs at $s$ and $t$. As usual, we have only recalled the definitions in the cases that are relevant to us.

Having introduced the notions above, the term $\sigma_{\mathcal{P}^-}(c)$ appeared in (A.16) is a $\mathbb{Z}/2$-valued function which depends only on the Morse index and the parity of the Maslov index, namely

$$\sigma_{\mathcal{P}^-}(c) = \sigma_{\mathcal{P}^-}(|\mu(c)|, ind(c)). \tag{A.17}$$

For a $Y_0$-vertex $c$, we shall modify the sub flow trees $\Gamma'_1$ and $\Gamma'_2$ above by cutting them a little bit earlier at the special punctures $s_1$ and $s_2$. There is an associated function $\sigma_0 : \mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ depending on the parity of Maslov indices of $s_1$ and $s_2$ such that

$$\sigma_{\mathcal{Y}_0}(c) = \sigma_0(|\mu(s_1)|, |\mu(s_2)|) + \eta + ord(\Gamma'_1) + ord(\Gamma'_2) + bm(\Gamma'_1) + bm(\Gamma'_2), \tag{A.18}$$

where

$$\begin{aligned} \eta = {} & e(\Gamma'_1) \cdot (bm(\Gamma'_2) + e(\Gamma'_2) + 1) + \dim FO_c(\Gamma'_1) \cdot (bm(\Gamma'_2) + |\mu(s_2)| + 1) \\ & + bm(\Gamma'_1) \cdot (|\mu(s_2)| + \dim FO_c(\Gamma'_2) + 1) + \dim \ker_a(\Gamma) + \dim \ker_{s_1}(\Gamma'_1) \\ & + \dim \ker_{s_2}(\Gamma'_2) + |\mu(s_1)| \cdot (1 + |\mu(s_2)| + bm(\Gamma'_2)). \end{aligned} \tag{A.19}$$

The case when $c$ is a 2-valent negative puncture can be regarded as the special case of a $Y_0$-vertex with one of the sub flow trees $\Gamma_1'$ and $\Gamma_2'$ being constant. Without loss of generality, we assume that $\Gamma_2'$ is constant. As before, we cut $\Gamma_1'$ a little bit earlier at the special puncture $s_1$ instead of $c$, so that the negative punctures of $\Gamma_1'$ are given by $b_1, \ldots, b_l$. If the punctures of $\Delta(\Gamma')$ are ordered as $s_1, b_1, \ldots, b_l$, let $tp(c) = 1$; if the punctures of $\Delta(\Gamma_1')$ are ordered as $b_1, \ldots, b_l, s_1$, let $tp(c) = 2$. There is a function $\sigma_1 : \mathcal{P}^2 \to \mathbb{Z}/2$ which depends on $|\mu(c)|$, $|\mu(s_1)|$ and $tp(c)$, with which the sign $\sigma_{\mathcal{P}^2}(c)$ has the form

$$\sigma_{\mathcal{P}^2}(c) = bm(\Gamma_1') \cdot |\mu(c)| + \dim \ker_{s_1}(\Gamma_1') + \sigma_1(c) + ord(\Gamma_1') + bm(\Gamma_1'). \tag{A.20}$$

Finally, for the positive puncture $a \in \mathcal{P}^+$, there is a number $\sigma_2(a) \in \mathbb{Z}/2$ which depends only on the Morse index and the parity of the Maslov index of $a$, namely

$$\sigma_2(a) = \sigma_2(|\mu(a)|, ind(a)). \tag{A.21}$$

By our assumption, there is no switch in $\Gamma$. In the case when the sub flow tree $\Gamma_1'$ has only one true vertex, the definition of $\sigma_{\mathcal{P}^+}(a)$ simplifies to

$$\sigma_{\mathcal{P}^+}(a) = \sigma_2(a) + (\dim \mathcal{U}(a) + 1)|\mu(a)|. \tag{A.22}$$

Having recalled the definitions of the individual signs $\nu_{int}, \nu_{end}$ and $\nu_{stab}$, the main result established in [58, 59] implies the following:

**Theorem A.1.1** (Karlsson)**.** *Let $\Lambda$ be a link of Legendrian spheres in $J^1(S)$, and assume that we have fixed all the initial orientation choices. Let $\mathcal{M}_\Lambda$ be the moduli space of rigid Morse flow trees determined by $\Lambda$. Then there is a coherent orientation on $\mathcal{M}_\Lambda$, with respect to which the sign $\varepsilon(\Gamma)$ of $\Gamma \in \mathcal{M}_\Lambda$ is given by*

$$\varepsilon(\Gamma) = \nu_{int}(\Gamma) \cdot \nu_{end}(\Gamma) \cdot \nu_{stab}(\Gamma). \tag{A.23}$$

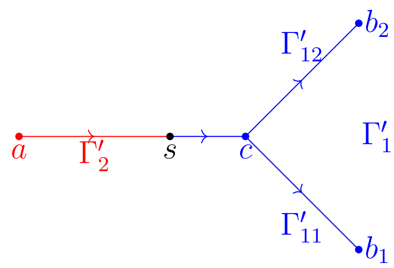


Figure A.1: The Morse flow tree $\Gamma$ and its associated sub flow trees $\Gamma'_1$ and $\Gamma'_2$

As an illustration, we compute here the signs of two elementary Morse flow trees in $\mathbb{R}^2$. The first example is a rigid Morse flow tree $\Gamma \in \mathcal{M}_\Lambda(a; b_1, b_2)$ which has a unique internal $Y_0$-vertex $c$, see Figure A.1. Furthermore, we assume that $a$ is a saddle point of some difference function $f_{ij} = f_i - f_j$, $b_1$ is a local minimum of some $f_{ik}$, while $b_2$ is a saddle point of some $f_{kj}$, where $f_i > f_k > f_j$.

In order to compute $\nu_{int}(\Gamma)$, we cut $\Gamma$ into two sub flow trees $\Gamma'_1$ and $\Gamma'_2$ at the special puncture $s$, where $s$ is contained in the same elementary region as $c$. Let $\ell_1 \subset \Gamma'_1$ be the edge which connects $s$ to $c$, and let $\ell_2 \subset \Gamma'_2$ be the edge that connects $a$ to $s$. As indicated in Figure A.1, the flow orientation of the sub flow tree $\Gamma'_2$ is fixed to be $\frac{\partial}{\partial x_1}$ for convenience.

The sub flow tree $\Gamma'_1$ contains two sub flow trees $\Gamma'_{11}$ and $\Gamma'_{12}$, where $\Gamma'_{11}$ is the $-\nabla f_{ik}$ flow line from $c$ to $b_1$, and $\Gamma'_{12}$ is the $-\nabla f_{kj}$ flow line from $c$ to $b_2$, see Figure A.1. Since the true vertex associated to $\Gamma'_{11}$ is $b_1 \in \mathcal{P}^-$, the flow-out $FO_c(\Gamma'_{11})$ is by definition the flow out of $b_1$ along the flow line $\Gamma'_{11}$. Similarly, $FO_c(\Gamma'_{12})$ is the flow-out of $b_2$ along $\Gamma'_{12}$. Choosing an interval $I_c$ centering at $c$ which is transverse to $\ell_1$, by our assumption that $b_2$ is a saddle point, it follows that

$$IM_c(\Gamma'_1) \cap I_c = \{c\}. \tag{A.24}$$

The flow-out $FO_s(\Gamma'_1)$ is then by definition $\ell_1 \cup \ell_2$. On the other hand, the associated true vertex of the sub flow tree $\Gamma'_2$ is $a \in \mathcal{P}^+$. Since $a$ is by assumption a saddle point, we have $FO_s(\Gamma'_2) = \ell_1 \cup \ell_2$.

Identify the tangent space $T_sS$ with $\mathbb{R}^2$ equipped with its standard orientation. By definition, $o(FO_c(\Gamma'_{11}))$ is the orientation of $T_c\mathcal{S}(b_1) \cong T_cS$, and $o(FO_c(\Gamma'_{12}))$ is the orientation of $T_c\mathcal{S}(b_2)$. This enables us to compute the orientation of the intersection manifold:

$$o(IM_c(\Gamma'_1)) := o_{int}(o(FO_c(\Gamma'_{11})), o(FO_c(\Gamma'_{12}))) = o(\Gamma'_{12}), \tag{A.25}$$

where $o(\Gamma'_{12})$ is the flow orientation of $\Gamma'_{12}$ at $c$. Since the true vertex $a$ of $\Gamma'_2$ is a saddle point, by our convention

$$o(FO_s(\Gamma'_2)) = o_{cap}(T_s\mathcal{U}(a)) = \frac{\partial}{\partial x_1}. \tag{A.26}$$

Since the standard domain $\Delta(\Gamma)$ has a unique slit, and $o(IM_c(\Gamma'_1)) \wedge o(FO_s(\Gamma'_2))$ gives the standard orientation on $T_sS$, we get from (A.15) that

$$\nu_{int}(\Gamma) = -1. \tag{A.27}$$

Note that if $\Gamma^{op} \in \mathcal{M}_\Lambda(a; b_2, b_1)$ is a rigid Morse flow tree obtained from $\Gamma$ by exchanging the labellings of $\Gamma'_{11}$ and $\Gamma'_{12}$, then $\nu_{int}(\Gamma^{op}) = \nu_{int}(\Gamma)$.

We now compute $\nu_{stab}(\Gamma)$. It follows immediately from our assumptions that

$$\sigma_{\mathcal{P}^-}(b_1) = \sigma_{\mathcal{P}^-}(|\mu(b_1)|, 0), \sigma_{\mathcal{P}^-}(b_2) = \sigma_{\mathcal{P}^-}(|\mu(b_2)|, 1), \tag{A.28}$$

and

$$\sigma_{\mathcal{P}^+}(a) = \sigma_2(|\mu(a)|, 1) + 2|\mu(a)|. \tag{A.29}$$

In order to determine $\sigma_{\mathcal{Y}_0}(c)$, consider the sub flow trees $\Gamma''_1$ and $\Gamma''_2$ of $\Gamma$ depicted as in Figure A.2. Since the standard domains $\Delta(\Gamma''_1)$ and $\Delta(\Gamma''_2)$ do not contain any boundary minimum, we have

$$bm(\Gamma''_1) = bm(\Gamma''_2) = 0, \tag{A.30}$$

and

$$ord(\Gamma''_1) = ord(\Gamma''_2) = 0. \tag{A.31}$$

By definition of true kernels, we have

$$\ker_a(\Gamma) = \ker_c(\Gamma'_{11}) \cap \ker_c(\Gamma'_{12}). \tag{A.32}$$

Since the true vertices associated to $\Gamma'_{11}$ and $\Gamma'_{12}$ are the negative punctures $b_1$ and $b_2$ respectively, we have

$$\ker_c(\Gamma'_{11}) = T_c FO_c(b_1), \ker_c(\Gamma'_{12}) = T_c FO_c(b_2). \tag{A.33}$$

Similarly,

$$\ker_{s_1}(\Gamma''_1) = T_{s_1} FO_{s_1}(b_1), \ker_{s_2}(\Gamma''_2) = T_{s_2} FO_{s_2}(b_2). \tag{A.34}$$

From the above we deduce that

$$\dim \ker_a(\Gamma) = 1, \dim \ker_{s_1}(\Gamma''_1) = 2, \dim \ker_{s_2}(\Gamma''_2) = 1, \tag{A.35}$$

and as a consequence,

$$\eta = 2(|\mu(b_2)| + 1) + 4 + |\mu(b_1)| \cdot (1 + |\mu(b_2)|), \tag{A.36}$$

where we have used the fact that

$$\mu(s_1) = \mu(b_1), \mu(s_2) = \mu(b_2). \tag{A.37}$$

Combining with (A.30) and (A.31), we get

$$\sigma_{\mathcal{Y}_0}(c) = \sigma_0(|\mu(b_1)|, |\mu(b_2)|) + |\mu(b_1)| \cdot (1 + |\mu(b_2)|). \tag{A.38}$$

By (A.16),

$$\nu_{stab}(\Gamma) = (-1)^{\sigma_0(|\mu(b_1)|,|\mu(b_2)|)+|\mu(b_1)|\cdot(1+|\mu(b_2)|)+\sigma_2(|\mu(a)|,1)+\sigma_{\mathcal{P}-}(|\mu(b_1)|,0)+\sigma_{\mathcal{P}-}(|\mu(b_2)|,1)}. \tag{A.39}$$

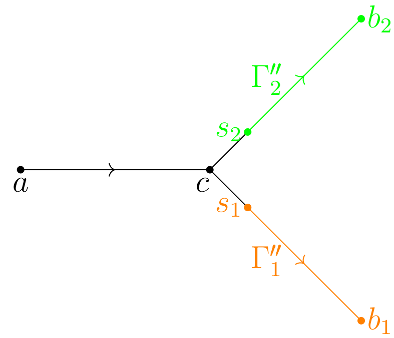


Figure A.2: The sub flow trees $\Gamma_1''$ and $\Gamma_2''$

Our second example deals with the case of a rigid Morse flow tree $\Xi \in \mathcal{M}_\Lambda(a; b, b^*)$ with a negative 2-valent puncture $b^*$, see the left-hand side of Figure A.3. We assume that the positive puncture $a$ is a maximum of some Morse function $f_{ij}$, the negative 1-valent puncture $b$ is a minimum of $f_{ik}$, and $b^*$ is a maximum of $f_{kj}$.

In this case, the sub flow tree $\Xi_2' \subset \Xi$ is a $-\nabla f_{ij}$ flow line from $a$ to $s$, and the sub flow tree $\Xi_1'$ inherits the 2-valent puncture $b^*$. Since the standard domain $\Delta(\Xi)$ still has a unique slit, the intersection orientation is determined by requiring that

$$\nu_{int}(\Xi) \cdot o(IM_{b*}(\Xi_1')) \wedge o(FO_s(\Xi_2')) \tag{A.40}$$

recovers the orientation of $T_sS$. To compute $o(IM_{b*}(\Xi_1'))$, consider the sub flow tree $\Xi_{11}' \subset \Xi_1'$, which is the $-\nabla f_{kj}$ flow line from $b^*$ to $b$. By definition, the flow out $FO_{b*}(\Xi_{11}')$ is the flow out of $b$ along $\Xi_{11}'$. Since $b$ is a minimum, we have by definition that

$$o(IM_{b*}(\Xi_1')) = o_{int}(o_{cap}(T_{b*}\mathcal{S}(b))), o_{cap}(T_{b*}\mathcal{U}(b^*)) = o(T_{b*}S). \tag{A.41}$$

On the other hand, the true vertex of $\Xi_2'$ is the positive puncture $a$, which shows that

$$o(FO_s(\Xi'_2)) = o(T_s\mathcal{U}(a)) = o(T_sS). \tag{A.42}$$

By (A.15), we have

$$\nu_{int}(\Xi) = 1. \tag{A.43}$$

The computation of $\nu_{stab}(\Xi)$ involves the determination of the individual signs $\sigma_{\mathcal{P}^+}(a)$, $\sigma_{\mathcal{P}^-}(b)$ and $\sigma_{\mathcal{P}^2}(b^*)$. Since $b$ is a minimum, we have

$$\sigma_{\mathcal{P}^-}(b) = \sigma_{\mathcal{P}^-}(|\mu(b)|, 0). \tag{A.44}$$

Similarly, since $a$ is a maximum, we get

$$\sigma_{\mathcal{P}^+}(a) = \sigma_2(|\mu(a)|, 2) + 3|\mu(a)|. \tag{A.45}$$

In order to compute $\sigma_{\mathcal{P}^2}(b^*)$, consider the sub flow tree $\Xi'_{11} \subset \Xi'_1$. We cut $\Xi'_{11}$ a little bit earlier so that it starts at a special puncture $s_1$ instead of $b^*$, and denote the resulting sub flow tree by $\Xi''_1$. Since the standard domain $\Delta(\Gamma'_{11})$ has no slit, we have

$$bm(\Xi''_1) = ord(\Xi''_1) = 0. \tag{A.46}$$

Since $b^* \in \mathcal{P}^2$, it follows that $\ker_{s_1}(\Xi''_1) = 0$, and

$$\sigma_{\mathcal{P}^2} = \sigma_1(b^*) = \sigma_1(|\mu(b^*)|, |\mu(b)|, 2). \tag{A.47}$$

In conclusion,

$$\nu_{stab}(\Xi) = (-1)^{\sigma_{\mathcal{P}^-}(|\mu(b)|,0)+\sigma_1(|\mu(b^*)|,|\mu(b)|,2)+\sigma_2(|\mu(a)|,2)+|\mu(a)|}. \tag{A.48}$$

Let $\Xi^{op} \in \mathcal{M}_\Lambda(a; b^*, b)$ be the rigid Morse flow tree whose fundamental domain $\Delta(\Xi^{op})$ has $b^*$ as its lower puncture, see the right-hand side of Figure A.3. In this case $tp(b^*) = 1$ and we conclude that $\varepsilon(\Xi) = -\varepsilon(\Xi^{op})$.

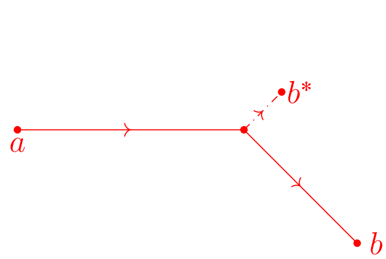


Figure A.3: The Morse flow trees $\Xi$ and $\Xi^{op}$, where the dotted edges are mapped to constant

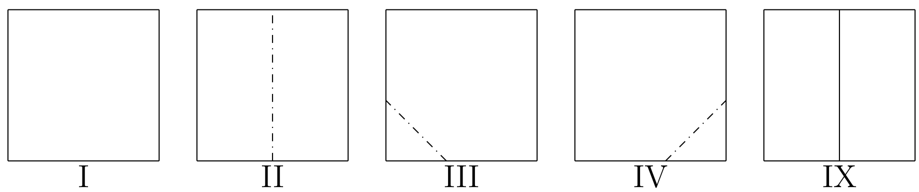


Figure A.4: The elementary squares of Types I, II, III, IV and IX

## A.2 Adaptation to the case of $\Lambda_{p,q,r}$

Recall that in [86], the cellular dg algebra $\mathcal{C}(\Lambda)$ of $\Lambda \subset J^1(S)$ is shown to be quasi-isomorphic to the Chekanov-Eliashberg dg algebra $CE^*(\Lambda)$ over $\mathbb{Z}/2$ by fixing a *transverse square decomposition* $\mathcal{E}_\pitchfork$ of $S$ and doing local analysis of Morse flow trees for each elementary square. In our specific case when $\Lambda = \Lambda_{p,q,r}$ with $p, q, r \geqslant 2$, it is not hard to see from Figure 6.4 that one can find a transverse square decomposition $\mathcal{E}_\pitchfork$ for $\Lambda_{p,q,r}$ such that only Type I, Type II, Type III, Type IV and Type IX squares appear in $\mathcal{E}_\pitchfork$ as elementary squares, see Figure A.4. From now on, fix such a transverse square decomposition.

Figure A.4 depicts the transverse square decomposition in a neighborhood of the 2-cell $e_7^2$ in the $\Lambda_{p,q,r}$-compatible polygonal decomposition of Figure 6.4, which consists of a single Type I square $\blacksquare_7 \cong [-1,1]^2$, and four Type II squares, two type III squares and two Type IV squares. Denote by $f_m : \blacksquare_7 \to \mathbb{R}$, the defining functions of the sheets $S_m$ above $\blacksquare_7$, which are labelled to satisfy $f_1 > \cdots > f_8$. Let $f_{mn} = f_m - f_n$ when $m < n$. Consider defining functions $f_m$ which are of the form

$$f_m(x_1, x_2) = h_m(x_1) + h_m(x_2), \tag{A.49}$$

where the $h_m$ are arranged so that $h_m - h_n$ has local minima at $-1$ and 1, and a single local maximum $\beta_{i,j} \in (-1, 1)$, such that

$$\beta_{1,2} < \beta_{1,3} < \cdots < \beta_{2,3} < \beta_{2,4} < \cdots < \beta_{7,8}. \tag{A.50}$$

The Reeb chords in $\blacksquare_7$ correspond to critical points of the functions $f_{mn}$. By abuse of notations, we shall denote these critical points by

$$a^{m,n}_{\pm,\pm}, b^{m,n}_U, b^{m,n}_R, b^{m,n}_D, b^{m,n}_L, c^{m,n}_7, \tag{A.51}$$

where $a^{m,n}_{\pm,\pm}$ are the four corners of $\blacksquare_7$, and they are minima of $f_{mn}$, $b^{m,n}_U, b^{m,n}_R, b^{m,n}_D, b^{m,n}_L$ are the saddle points of $f_{mn}$ located on the edges of $\blacksquare_7$, and $c^{m,n}_7$ is a local maximum of $f_{mn}$ lying in the interior of $\blacksquare_7$. By the analysis of [86], except for the $-\nabla f_{m,n}$ flow lines from $c^{m,n}_7$ to $b^{m,n}_U, b^{m,n}_R, b^{m,n}_D$ and $b^{m,n}_L$, there are four additional rigid Morse flow trees with positive puncture at $c^{m,n}_7$, see Figure A.6. The first two Morse flow trees have a 2-valent negative puncture at $c^{k,n}_7$ and $c^{m,k}_7$ respectively, while the last two Morse flow trees have a unique $Y_0$-vertex. This implies that

$$\begin{aligned} \partial c^{m,n}_7 &= b^{m,n}_U + b^{m,n}_L + b^{m,n}_R + b^{m,n}_D \\ &+ \sum_{m<k<n} a^{m,k}_{+,+} c^{k,n}_7 + \sum_{m<k<n} c^{m,k}_7 a^{k,n}_{-,-} + \sum_{m<k<n} b^{m,k}_U b^{k,n}_L + \sum_{m<k<n} b^{m,k}_R b^{k,n}_D \end{aligned} \tag{A.52}$$

in the Chekanov-Eliashberg algebra $CE^*(\Lambda_{p,q,r})$ over $\mathbb{Z}/2$. Notice that one can recover from (A.52) the formula (6.42) in the cellular dg algebra $\mathcal{C}(\Lambda_{p,q,r})$. In fact, this follows

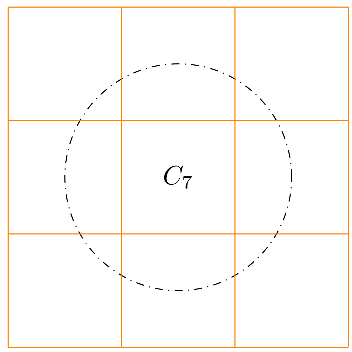


Figure A.5: The transverse square decomposition $\mathcal{E}_\pitchfork$ in a neighborhood of $e_7^2$

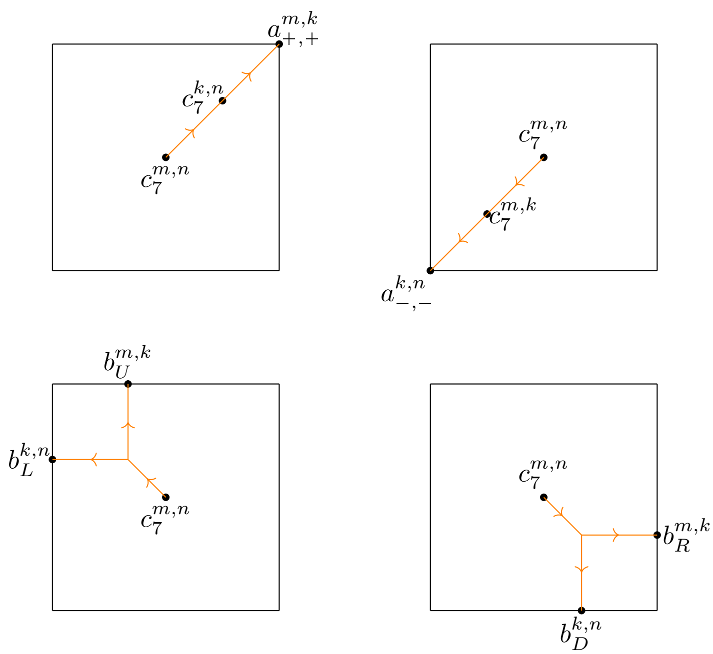


Figure A.6: Rigid Morse flow trees in a Type I square from the relations

$$b_L^{m,n} = b_R^{m,n} = b_D^{m,n} = 0, \tag{A.53}$$

$$a_{+,+}^{m,n} = a_{-,-}^{m,n}. \tag{A.54}$$

in the cellular dg algebra $\mathcal{C}_{||}(\Lambda_{p,q,r})$ associated to the cellular decomposition $\mathcal{E}_{||}$ obtained by shifting $p_x(\Sigma)$ into the borders of the elementary squares in the transverse square decomposition $\mathcal{E}_{\pitchfork}$, see Section 3.6 of [86]. In particular, the crossing arc corresponding to the 1-cell $e_1^1$ is shifted into the edges of $\blacksquare_7$. In fact, the same relations hold in the quotient dg algebra $\widetilde{CE}^*(\Lambda_{p,q,r})$ of $CE^*(\Lambda_{p,q,r})$ obtained by repeated applications of Lemma 5.3.1. This can be checked via the analysis of Morse flow trees with positive punctures at Reeb chords associated to 1-cells in $\mathcal{E}_{\pitchfork}$, which will be discussed below.

By (A.53) and (A.54), which also hold over an arbitrary field $\mathbb{K}$, we see that in order to determine $\partial c_7^{m,n}$ in $\widetilde{CE}^*(\Lambda_{p,q,r})$ over $\mathbb{K}$, it suffices to determine the signs of the flow line from $c_7^{m,n}$ to $b_U^{m,n}$, and the first two rigid Morse flow trees in Figure A.6. Denote by $\varepsilon_0(m,n)$, $\varepsilon_1(m,k,n)$ and $\varepsilon_2(m,k,n)$ the signs of these trees, we have in $\widetilde{CE}^*(\Lambda_{p,q,r})$ the formula

$$\begin{aligned}\partial c_7^{m,n} &= \varepsilon_0(m,n) b_1^{\sigma_0(m),\sigma_0(n)} + \sum_{m<k<n} \varepsilon_1(m,k,n) a_7^{\sigma_0(m),\sigma_0(k)} c_7^{k,n} \\ &\quad + \sum_{m<k<n} \varepsilon_2(m,k,n) c_7^{m,k} a_7^{\sigma_0(k),\sigma_0(n)},\end{aligned} \tag{A.55}$$

where we have changed our notations from $b_U^{m,n}$ and $a_{+,+}^{m,n} = a_{-,-}^{m,n}$ to $b_1^{\sigma_0(m),\sigma_0(n)}$ and $a_7^{\sigma_0(m),\sigma_0(n)}$, so that it coincides with the ones used in (6.42) for the cellular dg algebra. However, since we are dealing with the more general case of $\Lambda_{p,q,r}$ instead of $\Lambda_{2,2,2}$, the permutation $\sigma_0$ in the above formula is a composition of $(4,5)$ with a sequence of transpositions associated to the $A_{p-1}$-chain of unknots $\{\Lambda_{P_i}\}$, see Section 6.1.

The signs $\varepsilon_0(m,n)$, $\varepsilon_1(m,k,n)$ and $\varepsilon_2(m,k,n)$ can be explicitly computed using our discussions in Section A.1.

One can do similar analysis for the elementary squares $\blacksquare_{11}$ and $\blacksquare_{12}$ in $\mathcal{E}_{\pitchfork}$.

We also need to consider rigid Morse flow trees associated to generators corresponding to 1-cells. Let $e^1_\alpha \cong [-1,1]$ be a 1-cell in the transverse square decomposition $\mathcal{E}_{\pitchfork}$, with endpoints at the 0-cells $e^0_-$ and $e^0_+$. For each 0-cell $e^0_\beta$ in $\mathcal{E}_{\pitchfork}$, let $U(e^0_\beta)$ be a closed disc centered at $e^0_\beta$ with radius $\frac{1}{16}$. We can arrange so that there is a unique Reeb chord $a^{m,n}_\beta$ associated to every pair of sheets $S_m$ and $S_n$ with $z(S_m) > z(S_n)$, which corresponds to a minimum of $f_{mn}$, and the gradient $-\nabla f_{mn}$ points inward along $\partial U(e^0_\beta)$.

Let $U(e^1_\alpha)$ be a neighborhood of $e^1_\alpha$, which is depicted in Figure A.7 as the region bounded by the brown solid curve. It can be realized as a union $U(e^0_-) \cup \widehat{U}(e^1_\alpha) \cup U(e^0_+)$, where $\widehat{U}(e^1_\alpha)$ consists of a portion of $e^1_\alpha$ away from the endpoints together with parts of two elementary squares in $\mathcal{E}_{\pitchfork}$ which have $e^1_\alpha$ as their common boundary. We require that the boundary of $\widehat{U}(e^1_\alpha)$, which consists of a union of two paths $\gamma_1$ and $\gamma_2$, is contained within a distance of $\frac{1}{32}$ from $e^1_\alpha$, and both of $\gamma_1$ and $\gamma_2$ are parallel to $e^1_\alpha$.

In order to simplify the analysis of the rigid Morse flow trees in $U(e^1_\alpha)$, one needs to choose carefully the local defining functions $f_m$ of the sheets $S_m \subset \Lambda_{p,q,r}$, see Section 11 of [86] for details. For any point $(x_1,x_2) \in \gamma_i$ such that $f_{mn}(x_1,x_2) > 0$, we require that $-\nabla f_{mn}(x_1,x_2)$ is transverse to $\gamma_i$ at $(x_1,x_2)$ and is inward pointing, so that the Reeb chords in $U(e^1_\alpha)$ together with their differentials form a dg sub algebra of $CE^*(\Lambda_{p,q,r})$. The only Reeb chords in $\widehat{U}(e^1_\alpha)$ are:

- a Reeb chord with endpoints on the sheets $S_m$ and $S_n$ above $e^1_\alpha$, which corresponds to a saddle point $b^{m,n}_\alpha$ of $f_{mn}$, where $z(S_m) > z(S_n)$;

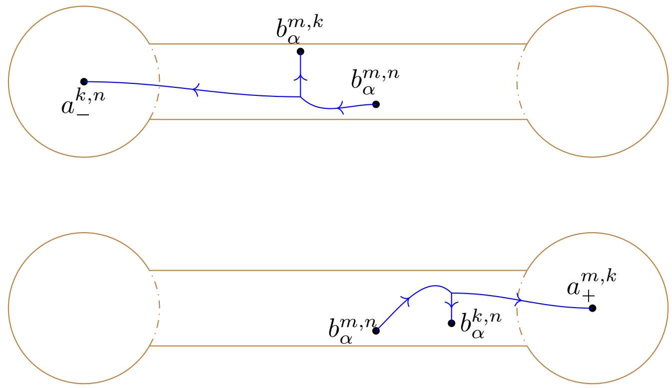


Figure A.7: Rigid Morse flow trees in neighborhoods of 1-cells

- if the sheets $S_m$ and $S_n$ cross above $e^1_\alpha$, then there is a Reeb chord $\tilde{b}^{n,m}_\alpha$.

With our choices of the defining functions $\{f_m\}$ of sheets $\{S_m\}$, one can further specify the locations of the Reeb chords $a^{m,n}_+, a^{m,n}_-, b^{m,n}_\alpha$ and $\tilde{b}^{n,m}_\alpha$. For details, we refer the readers to Section 5 of [86]. There are three types of rigid Morse flow trees with positive puncture at $b^{m,n}_\alpha$, namely the $-\nabla f_{mn}$ flow lines from $b^{m,n}_\alpha$ to $a^{m,n}_-$ and $a^{m,n}_+$; the Morse flow trees with one of its negative punctures at $\tilde{b}^{\bullet,\bullet}$; and the Morse flow trees described in Figure A.7, which have unique internal $Y_0$-vertices. This implies that

$$\partial b^{m,n}_\alpha = a^+_{m,n} + a^-_{m,n} + \sum_{m<k<n} a^{m,k}_+ b^{k,n}_\alpha + \sum_{m<k<n} b^{m,k}_\alpha a^{k,n}_- + x \qquad \text{(A.56)}$$

in $CE^*(\Lambda_{p,q,r})$ over $\mathbb{Z}/2$, where the term $x$ corresponds to those flow trees with one of their negative punctures at $\tilde{b}^{\bullet,\bullet}$. Note that if the sheets $S_k$ and $S_{k+1}$ meet at a cusp edge above $e^1_\alpha$, then $a^{k,k+1}_- = 1$ in the above formula, which corresponds to an $e$-vertex. By Lemma 8.5 of [86], we have $x = 0$ in the quotient dg algebra $\widetilde{CE}^*(\Lambda_{p,q,r})$. Again, what we have learned in Section A.1 enables us to determine the signs of the rigid Morse flow trees in Figure A.7, and we have

$$\begin{aligned}\partial b_\alpha^{m,n} &= \varepsilon_{\alpha,+}(m,n)a_{m,n}^{+} + \varepsilon_{\alpha,-}(m,n)a_{m,n}^{-} \\ &\quad + \sum_{m<k<n} \varepsilon_{\alpha,3}(m,k,n)a_+^{m,k}b_\alpha^{k,n} + \sum_{m<k<n} \varepsilon_{\alpha,4}(m,k,n)b_\alpha^{m,k}a_-^{k,n},\end{aligned} \tag{A.57}$$

where $\varepsilon_{\alpha,+}(m,n), \varepsilon_{\alpha,-}(m,n), \varepsilon_{\alpha,3}(m,k,n), \varepsilon_{\alpha,4}(m,k,n) \in \{\pm 1\}$ depend on $m, k$ and $n$.

The considerations above show that our computations of the cellular dg algebra $\mathcal{C}(\Lambda_{p,q,r})$ in Sections 6.2 and 6.3 already determine the Chekanov-Eliashberg algebra $CE^*(\Lambda_{p,q,r})$ over any field $\mathbb{K}$ up to signs in front of each of the monomials in the differentials of the generators of $\widetilde{CE}^*(\Lambda_{p,q,r})$. More precisely, denote by $\mathcal{G}^{\varepsilon}_{p,q,r}$ the dg algebra whose underlying graded associative $\Bbbk$-algebra structure is the same as $\mathcal{G}_{p,q,r}$, but whose differentials of the generators differ from that of $\mathcal{G}_{p,q,r}$ in the sense that there is a sign $(-1)^{\varepsilon_i}$ before every term appearing in the non-trivial differentials of the generators of $\mathcal{G}_{p,q,r}$. For example, the non-trivial differentials of the generators in the dg algebra $\mathcal{G}^{\varepsilon}_{2,2,2}$ are given by

$$da_1^* = (-1)^{\varepsilon_1}b_2c_2 + (-1)^{\varepsilon_2}b_3c_3, da_2^* = (-1)^{\varepsilon_3}b_1c_1 + (-1)^{\varepsilon_4}b_3c_3, \tag{A.58}$$

$$db_1^* = (-1)^{\varepsilon_5}c_1a_2, db_2^* = (-1)^{\varepsilon_6}c_2a_1, db_3^* = (-1)^{\varepsilon_7}c_3a_1 + (-1)^{\varepsilon_8}c_3a_2, \tag{A.59}$$

$$dc_1^* = (-1)^{\varepsilon_9}a_2b_1, dc_2^* = (-1)^{\varepsilon_{10}}a_1b_2, dc_3^* = (-1)^{\varepsilon_{11}}a_1b_3 + (-1)^{\varepsilon_{12}}a_2b_3, \tag{A.60}$$

$$dz_A = (-1)^{\varepsilon_{13}}a_1a_1^* + (-1)^{\varepsilon_{14}}a_2a_2^* + (-1)^{\varepsilon_{15}}c_1^*c_1 + (-1)^{\varepsilon_{16}}c_2^*c_2 + (-1)^{\varepsilon_{17}}c_3^*c_3, \tag{A.61}$$

$$dz_B = (-1)^{\varepsilon_{18}}a_1^*a_1 + (-1)^{\varepsilon_{19}}a_2^*a_2 + (-1)^{\varepsilon_{20}}b_1b_1^* + (-1)^{\varepsilon_{21}}b_2b_2^* + (-1)^{\varepsilon_{22}}b_3b_3^*, \tag{A.62}$$

$$\begin{aligned}dz_P &= (-1)^{\varepsilon_{23}}c_1c_1^* + (-1)^{\varepsilon_{24}}b_1^*b_1, dz_Q = (-1)^{\varepsilon_{25}}c_2c_2^* + (-1)^{\varepsilon_{26}}b_2b_2^*, \\ dz_R &= (-1)^{\varepsilon_{27}}c_3c_3^* + (-1)^{\varepsilon_{28}}b_3^*b_3.\end{aligned} \tag{A.63}$$

**Lemma A.2.1.** *Let $\mathbb{K}$ be any field, then there exists a vector*

$$\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_{2(p+q+r)+16}) \in (\mathbb{Z}/2)^{2(p+q+r)+16} \tag{A.64}$$

*such that $CE^*(\Lambda_{p,q,r})$ over $\mathbb{K}$ is quasi-isomorphic to the dg algebra $\mathcal{G}^{\boldsymbol{\varepsilon}}_{p,q,r}$.*

*Proof.* Since there is no swallowtail singularities in the Legendrian front of $\Lambda_{p,q,r}$, it follows from the arguments in Sections 5 and 6 of [86] that each term in the non-trivial differentials of the generators in $CE^*(\Lambda_{p,q,r})$ corresponds to a unique rigid Morse flow tree (instead of an odd number of them), and no two of the rigid Morse flow trees are cancelled in the differentials of $CE^*(\Lambda_{p,q,r})$ when $\mathbb{K} = \mathbb{Z}/2$. In particular, since $CE^*(\Lambda_{p,q,r})$ and $\mathcal{C}(\Lambda_{p,q,r})$ are related by a stable tame isomorphism, the same argument as in the proof of Proposition 6.2.1 applies when passing from $\mathcal{C}(\Lambda_{p,q,r})$ to $CE^*(\Lambda_{p,q,r})$, and from $\mathbb{Z}/2$ to an arbitrary field $\mathbb{K}$, which cancels all of the generators in $CE^*(\Lambda_{p,q,r})$ except for those in the quotient dg algebra $\mathcal{C}'(\Lambda_{p,q,r})$. This identifies the quotient dg algebra $\widetilde{CE}^*(\Lambda_{p,q,r})$ and $\mathcal{G}_{p,q,r}$ as $\Bbbk$-bimodules.

On the other hand, observe from our computations in Sections 6.2 and 6.3 that all of the generators in $\mathcal{C}(\Lambda_{p,q,r})$ associated to 0-cells are cancelled out in the quotient dg algebra $\mathcal{C}'(\Lambda_{p,q,r})$, therefore they do not contribute to $\widetilde{CE}^*(\Lambda_{p,q,r})$. Similarly, since all the generators of $\mathcal{C}(\Lambda_{p,q,r})$ associated to 2-cells can be cancelled out except for those in the matrices $C_7, C_{11}$ and $C_{12}$, in order to determine the dg algebra $\widetilde{CE}^*(\Lambda_{p,q,r})$ over $\mathbb{K}$, it suffices to use the formulae (A.55) (together with its analogues for the elementary squares $\blacksquare_{11}$ and $\blacksquare_{12}$) and (A.57). More precisely, let $A_i$ be the matrix of generators in the cellular dg algebra $\mathcal{C}(\Lambda_{p,q,r})$ associated to the 0-cell $e^0_i$. Since the cellular decomposition $\mathcal{E}_{||}$ is clearly finer than the one we used in Section 6.2 for computing $\mathcal{C}(\Lambda_{p,q,r})$, one can regard $A_i$ as the matrix of generators associated to a 0-cell $e^0_i$ in $\mathcal{E}_{||}$. Since there is no elementary squares of Types V, VI, VIII and XII in $\mathcal{E}_{\pitchfork}$, under the identification $\mathcal{C}_{||}(\Lambda_{p,q,r}) \cong CE^*(\Lambda_{p,q,r})/\mathcal{I}$ proved in Section 8.2 of [86], where $\mathcal{J}$ is the ideal generated by the exceptional generators in $CE^*(\Lambda_{p,q,r})$, it makes sense to talk about the matrix $A_i$ of generators in $CE^*(\Lambda_{p,q,r})$. After taking into account the orientation data of Morse flow trees, denote by $A_i^{\#}$ the corresponding matrix of generators in $CE^*(\Lambda_{p,q,r})$ over $\mathbb{K}$. Applying (A.57) iteratively, one can determine the value of $A_6^{\#}$ in the quotient dg algebra $\widetilde{CE}^*(\Lambda_{2,2,2})$, from which the differential $\partial b_7^{4,5}$ in $\widetilde{CE}^*(\Lambda_{p,q,r})$ can be computed. Use (A.57) again to determine the signs of rigid Morse flow trees with positive punctures at $b_7^{m,n}$, the value of $A_7^{\#}$ in $\widetilde{CE}^*(\Lambda_{p,q,r})$ can be computed, from which one gets the differentials of the generators in $C_7$ using (A.55). The same method can be applied to the Reeb chords above the elementary squares $\blacksquare_{11}$ and $\blacksquare_{12}$. This enables us to conclude that there is a term by term identification between the differentials of the generators in $\widetilde{CE}^*(\Lambda_{p,q,r})$ and the corresponding generators in $\mathcal{G}_{p,q,r}$, and the quasi-isomorphism $CE^*(\Lambda_{p,q,r}) \cong \mathcal{G}^{\varepsilon}_{p,q,r}$ follows. $\square$

**Proposition A.2.1.** *Let $\mathbb{K}$ be any field. There is a quasi-isomorphism*

$$CE^*(\Lambda_{p,q,r}) \cong \mathcal{G}_{p,q,r} \tag{A.65}$$

*between the Chekanov-Eliashberg algebra of $\Lambda_{p,q,r}$ and the Ginzburg algebra $\mathcal{G}_{p,q,r}$ over $\mathbb{K}$.*

*Proof.* Let $r = 2$. By Lemma A.2.1, there exists a vector $\boldsymbol{\varepsilon} \in (\mathbb{Z}/2)^{2(p+q)+20}$ such that $CE^*(\Lambda_{p,q,2}) \cong \mathcal{G}^{\varepsilon}_{p,q,2}$ as augmented dg algebras. On the other hand, it follows from Theorem 3.2.1 that

$$R\mathrm{Hom}_{\mathcal{G}^{\varepsilon}_{p,q,2}}(\Bbbk, \Bbbk) \cong \mathcal{V}_{p,q,2}, \tag{A.66}$$

where the $R$Hom on the left-hand side is taken with respect to the trivial augmentation on $\mathcal{G}^{\varepsilon}_{p,q,2}$. Since the Weinstein manifold $M_{p,q,2}$ can be regarded as the fiber of the double suspension of a Lefschetz fibration on $\mathbb{C}^2$, Corollary 3.1.1 of Section 3.1

can be applied to show that $\mathcal{V}_{p,q,2} = \mathcal{A}_{p,q,2} \oplus \mathcal{A}^\vee_{p,q,2}[-3]$. This determines completely the signs in $\boldsymbol{\varepsilon}$. For example, since

$$\mu^2_{\mathcal{V}_{p,q,2}}(c_1^\vee, b_1^\vee) = -(a_2^*)^\vee \tag{A.67}$$

and

$$\mu^2_{\mathcal{V}_{p,q,2}}(c_3^\vee, b_3^\vee) = (a_2^*)^\vee \tag{A.68}$$

in $\mathcal{V}_{p,q,2}$, we get $\varepsilon_3 = 0$ and $\varepsilon_4 = 1$, where as in Section 2.1, $g^\vee \in \mathcal{V}_{p,q,r}$ denotes the dual generator of $g \in \mathcal{G}_{p,q,r}$ under Koszul duality. The other signs can be similarly determined and it follows that $CE^*(\Lambda_{p,q,2}) \cong \mathcal{G}_{p,q,2}$ over $\mathbb{K}$.

Passing from $CE^*(\Lambda_{p,q,2})$ to the general case of $CE^*(\Lambda_{p,q,r})$ essentially requires a computation of the signs of rigid Morse flow trees determined by an $A_{r-2}$ link of unknots $\Lambda_{r-2}$. In fact, by the discussions above and the computations in Section 6.3, the only rigid Morse flow trees that contribute to the differentials of the additional generators in $\widetilde{CE}^*(\Lambda_{p,q,r})$ are those whose positive punctures correspond to Reeb chords which start at $\Lambda_{R_{j-1}}$ and end $\Lambda_{R_j}$ for some $j$. Let $c_{12}^{m,n}$ be the generator associated to such a Reeb chord. It follows from our assumptions that $b_{23}^{\sigma_0(m),\sigma_0(n)} = 0$ in $\widetilde{CE}^*(\Lambda_{p,q,r})$, so

$$\partial c_{12}^{m,n} = \sum_{m<k<n} \varepsilon_1(m,k,n) a_{14}^{\sigma_0(m),\sigma_0(k)} c_{12}^{k,n} + \sum_{m<k<n} \varepsilon_2(m,k,n) c_{12}^{m,k} a_{14}^{\sigma_0(k),\sigma_0(n)}. \tag{A.69}$$

Let $c_{12}^{m',n'}$ be the generator associated to the Reeb chord of the Legendrian unknot $\Lambda_{R_j}$, then from the configuration of the Legendrian front of an $A_{r-2}$ link of unknots, it is clear that $m' = n-1$. Consider the terms $a_{14}^{\sigma_0(m),\sigma_0(n-1)} c_{12}^{n-1,n}$ in $\partial c_{12}^{m,n}$ and $c_{12}^{n-1,n} a_{14}^{\sigma_0(n),\sigma_0(n')}$ in $\partial c_{12}^{n-1,n'}$, denote by $\Xi_1$ and $\Xi_2$ the corresponding rigid Morse flow trees. (A.57), together with our computations in Section A.1, shows that

$$a_{14}^{\sigma_0(m),\sigma_0(n-1)} = a_{14}^{\sigma_0(n),\sigma_0(n')} = b_{22}^{n-1,n} \tag{A.70}$$

in $\widetilde{CE}^*(\Lambda_{p,q,r})$. It follows that $\varepsilon(\Xi_1) = \varepsilon(\Xi)$ and $\varepsilon(\Xi_2) = \varepsilon(\Xi^{op})$, where $\Xi$ and $\Xi^{op}$ are the rigid Morse flow trees depicted in Figure A.3. The computation at the end of Section A.1 then allows us to conclude that $\varepsilon(\Xi_1) = -\varepsilon(\Xi_2)$. One can make the choice of capping orientations so that $\varepsilon(\Xi_1) = -1$ and $\varepsilon(\Xi_2) = 1$.

Using the notations as in the proof of Proposition 8.1.2, we have proved that

$$\partial z_{R_j} = z_{j-1}^* z_j - z_j z_{j+1}^* \tag{A.71}$$

in $\widetilde{CE}^*(\Lambda_{p,q,r})$. Moreover, the additional term $z_1 z_1^*$ in the differential of $z_{Q_1}$ has negative sign, since its associated rigid Morse flow tree has the same sign as $\Xi$. This proves the quasi-isomorphism $\widetilde{CE}^*(\Lambda_{p,q,r}) \cong \mathcal{G}_{p,q,r}$. □

# Appendix B

# The cellular dg algebra of $\Lambda_{1,1,0}$

We compute the Chekanov-Eliashberg dg algebra $CE^*(\Lambda_{1,1,0})$ of the Legendrian surface $\Lambda_{1,1,0} \subset \mathbb{R}^5$ over $\mathbb{K} = \mathbb{Z}/2$ mentioned in Section 6.4 in this appendix, so as to give a proof of our claim (6.176). To do this, we need to use Proposition 5.4.1 in the case when $\Lambda_K$ possess a cone singularity, so we start by describing how to modify the cellular dg algebra $\mathcal{C}(\Lambda)$ when a cone singularity appears in the front projection of $\Lambda$.

## B.1 Cone singularities

Although we will not deal with swallowtail singularities of 2-dimensional Legendrian fronts in this thesis, there is one special case which involves swallowtail singularities that will be of interest to us in this appendix, namely when several swallowtail singularities can be grouped together, to form a so-called *cone singularity*. Such kind of singularities appear naturally when doing front spin of a Legendrian arc, see Section 5.4. Notice that the appearance of a cone singularity in the Legendrian front means that the front projection is *not* generic, so modifications to the definition in Section 5.1 are required.

Let $\Lambda \subset J^1(S)$ be a Legendrian surface with generic base and front projections except for the presence of finitely many cone points whose base projections are disjoint from the images of the cusps and crossings in $p_{x,z}(\Lambda)$. We can associate a dg algebra

$$(\mathcal{C}^c(\Lambda), d^c_{\mathcal{C}}) \tag{B.1}$$

to $\Lambda$ using a $p_x(\Lambda)$-compatible polygonal decomposition as in Section 5.1, but with the following modifications at a cone point.

Assume that the cone point projects to the 0-cell $e^0_\alpha$ under $p_x$. In a small neighbourhood of $e^0_\alpha$, the sheets of $\Lambda$ are labelled as $S_1, \dots, S_r$ according to the decreasing order of their $z$-coordinates, and the cone point connects the sheets $S_k$ and $S_{k+1}$ for some $1 \leqslant k \leqslant r-1$. The Maslov potential $\mu$ satisfies

$$\mu(S_k) = \mu(S_{k+1}) + 1. \tag{B.2}$$

For the generators $a^{m,n}_\alpha$ associated to the 0-cell $e^0_\alpha$, we require that $a^{k,k+1}_\alpha = 0$. The same convention will be taken when computing differentials on the generators associated to 1-cells or 2-cells which have $e^0_\alpha$ as their initial or terminal vertices.

Let $e^2_\beta$ be a 2-cell which borders $e^0_\alpha$. When computing $d^c_{\mathcal{C}} C_\beta$ for the 2-cell $e^2_\beta$, we add an additional edge to $\partial e^2_\beta$ at this vertex, and insert the matrix

$$U_\alpha = E + \sum_{i<k} a^{i,k+1}_\alpha \Delta_{i,k} + \sum_{k+1<i} a^{k,i}_\alpha \Delta_{k+1,i} \tag{B.3}$$

into the product of edges that appears in $d_{\mathcal{C}} C_\beta$, see (5.15). Here $\Delta_{i,j} = (\delta_{ij})$. More precisely, under the notations in (5.15), if $e^0_\alpha$ is the 0-cell associated to the base projection of the cone singularity which connects the adjacent 1-cells $e^1_{\beta,m}$ and $e^1_{\beta,m+1}$ in $c^2_\beta \circ \gamma_+$, with $1 \leqslant m < m+1 \leqslant n_+$, then $d_{\mathcal{C}} C_\beta$ takes the following form:

$$\begin{aligned} d_{\mathcal{C}} C_\beta &= A_{\beta,+} C_\beta + C_\beta A_{\beta,-} \\ &\quad + (E + B_{\beta,n_+})^{\varepsilon_{n_+}} \cdots (E + B_{\beta,m+1})^{\varepsilon_{m+1}} U_\alpha (E + B_{\beta,m})^{\varepsilon_m} \cdots (E + B_{\beta,1})^{\varepsilon_1} \\ &\quad + (E + B_{\beta,n_+ + n_-})^{\varepsilon_{n_+ + n_-}} \ldots (E + B_{\beta,n_+ + 1})^{\varepsilon_{n_+ + 1}}. \end{aligned} \tag{B.4}$$

Note that after a Legendrian isotopy, we can achieve that the resulting Legendrian surface $\Lambda'$ has generic front and base projections, and the front projection $p_{x,z}(\Lambda')$ will then involve 4 swallowtail singularities, see Section 3.1 of [31]. In particular, there is a well-defined cellular dg algebra $\mathcal{C}(\Lambda')$ in the usual sense. It is proved in [85] that there is a quasi-isomorphism

$$\mathcal{C}^c(\Lambda) \cong \mathcal{C}(\Lambda'). \tag{B.5}$$

However, as we shall see in Section B.2, working with cone singularities simplifies the computation of $\mathcal{C}(\Lambda')$, for this reason we will keep the cone points in the front projections instead of resolving them.

## B.2 Computation of $\mathcal{C}(\Lambda_{1,1,0})$

We compute the cellular dg algebra for the Legendrian surface $\Lambda_{1,1,0} \subset J^1(\mathbb{R}^2)$, whose front projection is given in Figure 6.7, where the thick dot represents a cone singularity. Note that the front of $\Lambda_{1,1,0}$ can be realized as a spin of a Legendrian arc $K_{1,1,0} \subset \mathbb{R}^3$, whose associated cellular decomposition is shown in Figure B.1. As is explained in Section 4.1 of Casals-Murphy [22], $\Lambda_{1,1,0}$ is the Legendrian attaching sphere which defines the Weinstein manifold $M_{1,1,0}$.

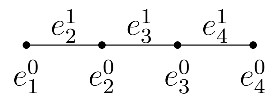


Figure B.1: Cellular decomposition associated to $K_{1,1,0}$

The matrices associated to the 0-cells are denoted correspondingly by $A_1, A_2, A_3$ and $A_4$; similarly, the matrices associated to the 1-cells are $B_2, B_3$ and $B_4$.

**Proposition B.2.1.** *The cellular dg algebra* $\mathcal{C}(\Lambda_{1,1,0})$ *is quasi-isomorphic to a dg algebra with generators*

$$b_3^{1,2}, \check{b}_4^{1,2}, \check{b}_4^{2,4}, \tag{B.6}$$

*and gradings*

$$|b_3^{1,2}| = -2, |\check{b}_4^{1,2}| = 1, |\check{b}_4^{2,4}| = -2, \tag{B.7}$$

*whose differentials are given by*

$$d_{\mathcal{C}} b_3^{1,2} = d_{\mathcal{C}} \check{b}_4^{1,2} = 0, d_{\mathcal{C}} \check{b}_4^{2,4} = b_3^{1,2} \check{b}_4^{1,2} + \check{b}_4^{1,2} b_3^{1,2}. \tag{B.8}$$

*Proof.* We start by computing the cellular dg algebra $\mathcal{C}(K_{1,1,0})$. Since there is only a cusp edge above the 0-cell $e_1^0$, $A_1 = N$ is a $2 \times 2$ nilpotent block. Since there is no crossing above the 1-cell $e_2^1$, we see that the only non-zero entries in $A_2$ are

$$a_2^{1,4} = a_2^{2,3} = 1. \tag{B.9}$$

Above $e_3^0$, there is a crossing between the first and second strands in $K_{1,1,0}$, which shows that $a_3^{1,2} = 0$. Using (5.12), one can cancel $a_3^{2,3}$ with $b_3^{2,3}$, and $a_3^{1,3}$ with $b_3^{1,3}$, so that only $b_3^{1,2}$ remains in $B_3$ as a generator of $\mathcal{C}'(K_{1,1,0})$. From the formula of $d_{\mathcal{C}} B_3$, it follows that in $\mathcal{C}'(K_{1,1,0})$,

$$A_3 = \begin{bmatrix} 0 & 0 & b_3^{1,2} & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.10}$$

Similarly, the crossing between the third and fourth strands above $e_4^0$ leaves us with the generator $b_4^{3,4}$ in $B_4$, based on which the matrix $A_4$ after cancelling generators can also be determined, i.e.

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & b_4^{3,4} \\ 0 & 0 & b_3^{1,2} & 1 + b_3^{1,2} b_4^{3,4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{B.11}$$

By Proposition 5.4.1, the dg algebra $\mathcal{C}(\Lambda_{1,1,0})$ can be obtained by suspending $\mathcal{C}(K_{1,1,0})$ relative to the dg subalgebra $\mathcal{O}(K_{1,1,0})$, with the additional modification that

$$D(a_4^{1,3}) = b_4^{3,4}, D(a_4^{2,3}) = 1 + b_3^{1,2} b_4^{3,4}. \tag{B.12}$$

From the formula of $d_{\mathcal{C}} B_3$, we see that the only remaining generator in $\check{B}_3$ is $\check{b}_3^{1,2}$. Using the formula for $d_{\mathcal{C}} \check{B}_4$ we compute

$$d_{\mathcal{C}} \check{b}_4^{1,2} = d_{\mathcal{C}} \check{b}_4^{3,4} = 0, \tag{B.13}$$

and

$$d_{\mathcal{C}} \check{b}_4^{1,3} = b_4^{3,4} + \check{b}_4^{1,2} b_3^{1,2}, \tag{B.14}$$

$$d_{\mathcal{C}} \check{b}_4^{1,4} = (b_4^{3,4})^2 + \check{b}_4^{3,4} + \check{b}_4^{1,2}, \tag{B.15}$$

$$d_{\mathcal{C}} \check{b}_4^{2,3} = 1 + \check{b}_3^{1,2} + b_3^{1,2} b_4^{3,4}, \tag{B.16}$$

$$d_{\mathfrak{C}}\check{b}_4^{2,4} = (1 + b_3^{1,2} b_4^{3,4}) b_4^{3,4} + b_3^{1,2} \check{b}_4^{3,4}. \tag{B.17}$$

These computations imply that the generators

$$(\check{b}_4^{1,3}, b_4^{3,4}), (\check{b}_3^{1,2}, \check{b}_4^{2,3}), (\check{b}_4^{1,4}, \check{b}_4^{3,4}) \tag{B.18}$$

can be cancelled in pair.

One can equip $\Lambda_{1,1,0}$ with a Maslov potential $\mu_{1,1,0} : \Lambda_{1,1,0} \to \mathbb{Z}$ as indicated in Figure 6.7. Note that for the values of $\mu_{1,1,0}$ on the sheets connected by the cone singularity, we have followed the grading convention (B.2). This enables us to deduce the gradings in (B.7). $\square$

# Bibliography


[1] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. 59 (2006), no. 2, 254-316.

[2] M. Abouzaid, *A geometric criterion for generating the Fukaya category*, Publ. Math. IHES 112 (2010), 191-240.

[3] M. Abouzaid, *A topological model for the Fukaya categories of plumbings*, J. Differential Geom. 87 (2011), no. 1, 1-80.

[4] M. Abouzaid, *On wrapped Fukaya category and based loops*, J. Symplectic Geom. 10 (2012), no. 1, 27-79.

[5] M. Abouzaid, *Symplectic cohomology and Viterbo's theorem*, in *Free loop spaces in geometry and topology*, volume 24 of IRMA Lect. Math. Theor. Phys., pages 271-485. Eur. Math. Soc., Zurich, 2015.

[6] M. Abouzaid, D. Auroux, and L. Katzarkov, *Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces*, Publ. Math. IHES 123 (2016), 199-282.

[7] M. Abouzaid and P. Seidel, *An open string analogue of Viterbo functoriality*, Geom. Topol. 14 (2010), no. 2, 627-718.

[8] M. Abouzaid, I. Smith, *Exact Lagrangians in plumbings*, Geom. Funct. Anal. 22 (2012), no. 4, 785-831.

[9] J.F. Adams, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 409-412.

[10] J.F. Adams and P.J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. 30 (1956), 305-330.

[11] L. Avramov, *(Contravariant) Koszul duality for dg algebras*. In: A. Buan, I. Reiten, Ø. Solberg, (eds) Algebras, Quivers and Representations. Abel Symposia, vol 8. Springer, Berlin, Heidelberg (2013).

[12] A. Beilinson, *Coherent sheaves on* $\mathbb{P}^n$ *and problems in linear algebra*, Funktsional, Anal. i Prilozhen. 12 (1978), no. 3, 68-69.

[13] A. Beilinson, V. Ginzburg and W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), no. 2, 473-527.

[14] C. Bessenrodt, T. Holm, A. Zimmermann, *Generalized Reynolds ideals for nonsymmetric algebras*, J. Algebra 312, 985-994 (2007).

[15] A.-L. Biolley, *Floer homology, symplectic and complex hyperbolicities*, arXiv:math.SG/0404551.

[16] F. Bourgeois, T. Ekholm, Y. Eliashberg, *Effect of Legendrian surgery*, With an appendix by S. Ganatra and M. Maydanskiy, Geom. Topol., 16, (2012), 301-389.

[17] F. Bourgeois, T. Ekholm, and Y. Eliashberg, *Symplectic homology product via Legendrian surgery*, Proc. Natl. Acad. Sci. USA, 108(20):8114-8121, 2011.

[18] F. Bourgeois and A. Oancea, *The Gysin exact sequence for $S^1$-equivariant symplectic homology*, J. Topol. Anal., 5(4):361-407, 2013.

[19] F. Bourgeois and A. Oancea, *$S^1$-equivariant symplectic homology and linearized contact homology*, Int. Math. Res. Not., (13):3849-3937, 2017.

[20] F. Bourgeois and A. Oancea, *An exact sequence for contact- and symplectic homology*, Invent. Math., 175(3):611-680, 2009.

[21] N. A'Campo, *Le groupe de monodromie du deploiement des singularites isolees de courbes planes I*, Mathematische Annalen, 213 (1975), 1-32.

[22] R. Casals and E. Murphy, *Legendrian fronts for affine varieties*, Duke Math. J. Volume 168, Number 2 (2019), 225-323.

[23] B. Chantraine, G. Dimitroglou-Rizell, P. Ghiggini, and R. Golovko, *Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors*, arXiv:1712.09126.

[24] C. Cibils, E. Marcos, M.J. Redondo and A. Solotar, *Cohomology of split algebras and of trivial extensions*, Glasgow Math. J. 45 (1) (2003) 21-40.

[25] K. Cieliebak and J. Latschev, *The role of string topology in symplectic field theory*, New perspectives and challenges in symplectic field theory, CRM Proc. Lecture Notes, vol. 49, Amer. Math. Soc., Providence, RI, 2009, pp. 113-146.

[26] M. Chas and D. Sullivan, *String topology*, arXiv:math/9911159.

[27] Y. Chekanov, *Differential algebra of Legendrian links*, Invent. Math., 150 (2002), no. 3, 441-483.

[28] X. Chen, S. Yang and G. Zhou, *Batalin-Vilkovisky algebras and the noncommutative Poincare duality of Koszul Calabi-Yau algebras*, Journal of Pure and Applied Algebra 220 (2016) 2500-2532.

[29] R. Cohen and S. Ganatra, *Calabi-Yau categories, the Floer theory of a cotangent bundle, and the string topology of the base*, preliminary version available at http://math.stanford.edu/ ralph/papers.html.

[30] B. Davison, *Superpotential algebras and manifolds*, Adv. Math. 231(2), 879-912 (2012).

[31] G. Dimitroglou Rizell, *Knotted Legendrian surfaces with few Reeb chords*, Algebr. Geom. Topol. 11 (2011) 2903-2936.

[32] T. Ekholm, *Morse flow trees and Legendrian contact homology in 1-jet spaces*, Geom. Topol., 11:1083-1224, 2007.

[33] T. Ekholm, *Holomorphic curves for Legendrian surgery*, arXiv:1906.07228.

[34] T. Ekholm, J. Etnyre, and M. Sullivan, *The contact homology of Legendrian submanifolds in* $\mathbb{R}^{2n+1}$, J. Differential Geom., 71(2):177-305, 2005.

[35] T. Ekholm, K. Honda and T. Kálmán, *Legendrian knots and exact Lagrangian cobordisms*, J. Eur. Math. Soc. Volume 18, Issue 11, 2016, pp. 2627-2689.

[36] T. Ekholm and Y. Lekili, *Duality between Lagrangian and Legendrian invariants*, arXiv:1701.01284.

[37] T. Ekholm and L. Ng, *Legendrian contact homology in the boundary of a subcritical Weinstein 4-manifold*, J. Differential Geom. 101 (2015), no. 1, 67-157.

[38] Y. Eliashberg, *Weinstein manifolds revisited*, arXiv:1707.03442.

[39] Y. Eliashberg, A. Givental and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal. 2000, Special Volume, Part II, 560-673.

[40] T. Etgü and Y. Lekili, *Koszul duality patterns in Floer theory*, Geometry and Topology 21 (2017) 3313-3389 .

[41] T. Etgü and Y. Lekili, *Fukaya categories of plumbings and multiplicative pre-projective algebras*, arXiv:1703.04515, to appear in Quantum Topology.

[42] K. Fukaya, *Cyclic symmetry and adic convergence in Lagrangian Floer theory*, Kyoto J. Math. Volume 50, Number 3 (2010), 521-590.

[43] S. Ganatra, *Cyclic homology, $S^1$-equivariant Floer cohomology, and Calabi-Yau structures*, arXiv:1912.13510.

[44] S. Ganatra, *Symplectic cohomology and duality for the wrapped Fukaya category*, arXiv:1304.7312.

[45] S. Ganatra, T. Perutz, and N. Sheridan, *Mirror symmetry: from categories to curve-counts*, arXiv:1510.03839.

[46] S. Ganatra, J. Pardon and V. Shende, *Covariantly functorial Floer theory on Liouville sectors*, arxiv:1706.03152.

[47] S. Ganatra, J. Pardon and V.Shende, *Structural results in wrapped Floer theory*, arXiv:1809.03427.

[48] S. Ganatra and D. Pomerleano, *A Log PSS morphism with applications to Lagrangian embeddings*, arXiv:1611.06849.

[49] S. Ganatra and D. Pomerleano, *Symplectic cohomology rings of affine varieties in the topological limit*, arXiv:1811.03609.

[50] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math.AG/0612139.

[51] E. Giroux and J. Pardon, *Existence of Lefschetz fibrations on Stein and Weinstein domains*, arXiv:1411.6176.

[52] R. Golovko, *A note on Lagrangian cobordisms between Legendrian submanifolds of* $\mathbb{R}^{2n+1}$, Pacific J. Math., 261(1):101-116, 2013.

[53] M. Gross, P. Hacking and S.Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, Publications mathematiques de l'IHES, November 2015, Volume 122, Issue 1, 65-168.

[54] J. Gutt and M. Hutchings, *Symplectic capacities from positive* $S^1$*-equivariant symplectic homology*, Algebr. Geom. Topol., 18(6):3537-3600, 2018.

[55] D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 120 (575) (1996).

[56] D. Hermann, *Holomorphic curves and hamiltonian systems in an open set with restricted contact type boundary*, Duke Math. J. 103, 335-374, (2000).

[57] E. Herscovich, *Hochschild (co)homology and Koszul duality*, arXiv: 1405.2247.

[58] C. Karlsson, *Orientations of Morse flow trees in Legendrian contact homology*, arXiv:1601.07346.

[59] C. Karlsson, *To compute orientations of Morse flow trees in Legendrian contact homology*, arXiv:1704.05156.

[60] A. Keating, *Lagrangian tori in four-dimensional Milnor fibers*, Geom. Funct. Anal. Vol. 25 (2015), 1822-1901.

[61] A. Keating, *Homological mirror symmetry for hypersurface cusp singularities*, A. Sel. Math. New Ser. (2017), 1-42.

[62] B. Keller, *Deformed Calabi-Yau completions*, with an appendix by M. Van den Bergh, J. Reine Angew. Math. 654 (2011), 125-180.

[63] B. Keller, *Erratum to "Deformed Calabi-Yau completions"*, arXiv:1809.01126.

[64] B. Keller, *Derived invariance of higher structures on the Hochschild complex*, preprint (2003).

[65] B. Keller, *Deriving DG categories*, Ann. Sci. Ec. Norm. Sup. 27 (1994), 63-102.

[66] B. Keller and P. Nicolás, *Weighted structures and simple dg modules for positive dg algebras*, International Mathematics Research Notices, Vol. 2013, No. 5, pp. 1028-1078, 2013.

[67] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson-Thomas invariants and cluster transformations*, arXiv:0811.2435.

[68] M. Kontsevich and Y. Soibelman, *Notes on $A_\infty$-algebras, $A_\infty$-categories and non-commutative geometry*, Homological mirror symmetry, 2009, 153-219.

[69] M. Kontsevich and Y. Vlassopoulos, *Pre-Calabi-Yau algebras and topological quantum field theories*, in Preparation.

[70] Y. Lekili and A. Polishchuk, *Homological mirror symmetry for higher dimensional pairs of pants*, arXiv:1811.04264, to appear in Compos. Math.

[71] Y. Lekili, A. Polishchuk, *Arithmetic mirror symmetry for genus 1 curves with n marked points*, Selecta Math. (N.S.) 23 (2017), 1851-1907.

[72] Y. Lekili and K. Ueda, *Homological mirror symmetry for Milnor fibers via moduli of $A_\infty$ structures*, arXiv:1806.04345.

[73] H. Lenzing and A. Skowronski, *Quasi-tilted algebras of canonical type*, Colloq. Math. 71 (1996) 161-181.

[74] Y. Li, *Koszul duality via suspending Lefschetz fibrations*, J. Topology (2019) 12 (4), 1174-1245.

[75] Y. Li, *Exact Calabi-Yau categories and disjoint Lagrangian spheres*, arXiv:1907.09257.

[76] J.L. Loday, *Cyclic homology*, second ed., Grundlehren der Mathematischen Wissenschaften, Vol. 301, Springer, Berlin, (1998).

[77] D. Lu, J. Palmieri, Q. Wu and J. Zhang, *Koszul equivalences in $A_\infty$ algebras*, New York J. Math. 14 (2008), 325-378.

[78] V.A. Lunts, *Categorical resolution of singularities*, J. Algebra 323 (2010), no. 10, 2977-3003.

[79] M. McLean, *Symplectic invariance of uniruled affine varieties and log Kodaira dimension*, Duke Math. J. Volume 163, Number 10 (2014), 1929-1964.

[80] M. McLean, *Lefschetz fibrations and symplectic homology*, Geom. Topol. 13 (2009), 1877-1944.

[81] D. Nadler, *A combinatorial calculation of the Landau-Ginzburg model $M = \mathbb{C}^3, W = z_1z_2z_3$*, Selecta Math. 23 (2017), no. 1, 519-532.

[82] D. Nadler and E. Zaslow, *Constructible Sheaves and the Fukaya Category*, J. Amer. Math. Soc. 22, 233-286, (2009).

[83] J. Pascaleff, *On the symplectic cohomology of log Calabi-Yau surfaces*, arXiv:1304.5298, to appear in Geometry and Topology.

[84] D. Pomerleano, *Curved String Topology and Tangential Fukaya Categories II*, arXiv:1208.4881.

[85] D. Rutherford and M. Sullivan, *Cellular Legendrian contact homology for surfaces, Part I*, arXiv:1608.02984.

[86] D. Rutherford and M. Sullivan, *Cellular Legendrian contact homology for surfaces, Part II*, arXiv:1608.03011.

[87] T. Schedler, *Zeroth Hochschild homology of preprojective algebras over the integers*, Advances in Mathematics, Vol. 299 (2016), 451-542.

[88] E. Segal, *The $A_\infty$ deformation theory of a point and the derived categories of local Calabi-Yaus*, J. Algebra 320 (2008), no. 8, 3232-3268.

[89] P. Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zürich Lect. in Adv. Math., European Math. Soc., Zürich, 2008.

[90] P. Seidel, *Lectures on categorical dynamics and symplectic topology*, available at http://www-math.mit.edu/ seidel/.

[91] P. Seidel, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France 128 (2000), no. 1, 103-149.

[92] P. Seidel, *Suspending Lefschetz fibrations, with an application to local mirror symmetry*, Comm. Math. Phys. 297 (2010), 515-528.

[93] P. Seidel, *$A_\infty$-subalgebras and natural transformations*, Homology, Homotopy Appl., 10:83-114, 2008.

[94] P. Seidel, *A biased view of symplectic cohomology*, Current developments in mathematics, International Press, Somerville, MA, 2008, 211-253.

[95] P. Seidel, *Picard-Lefschetz theory and dilating $\mathbb{C}^*$-actions*, J. Topology (2015) 8 (4): 1167-1201.

[96] P. Seidel, *Disjoinable Lagrangian spheres and dilations*, Invent math (2014) 197:299-359.

[97] P. Seidel, *Fukaya $A_\infty$-structures associated to Lefschetz fibrations*, J. Symplectic Geom. Volume 10, Number 3 (2012), 325-388.

[98] P. Seidel, *Lagrangian homology spheres in $(A_m)$ Milnor fibres via $\mathbb{C}^*$-equivariant $A_\infty$-modules*, Geom. Topol. 16:2343-2389, (2012).

[99] P. Seidel, *Fukaya categories and deformations*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 2002, pp. 351-360.

[100] P. Seidel, *The equivariant pair-of-pants product in fixed point Floer cohomology*, Geom. Funct. Anal. 25 (2015), no. 3, 942-1007.

[101] P. Seidel, *A long exact sequence for symplectic Floer cohomology*, Topology 42 (2003), no. 5, 1003-1063.

[102] P. Seidel and I. Smith, *The symplectic topology of Ramanujam's surface*, Comment. Math. Helv. 80, 859-881 (2005).

[103] P. Seidel and J. Solomon, *Symplectic cohomology and q-intersection numbers*, Geom. Funct. Anal. 22, 443-477 (2012).

[104] S. Sivek, *A bordered Chekanov-Eliashberg algebra*, J. Topol. 4(1), 73-104 (2011).

[105] I. Smith, *Quiver algebras as Fukaya categories*, Geom. Topol. 19 (2015), no. 5, 2557-2617.

[106] I. Smith and R. Thomas, *Symplectic surgeries from singularities*, Turkish J. Math. 27 (2003), 231-250.

[107] *Stacks Project*, available at: http://stacks. math. columbia. edu.

[108] T. Tom Dieck and T. Petrie, *Contractible affine surfaces of Kodaira dimension one*, Japan. J. Math. (N.S.) 16 (1990), 147-169.

[109] D. Tonkonog, *From symplectic cohomology to Lagrangian enumerative geometry*, Adv. Math. 352, 717-776, 2019.

[110] T. Tradler, *The Batalin-Vilkovisky algebra on Hochschild cohomology induced by infinity inner products*, Ann. Inst. Fourier 58 (2008), no. 7, 2351-2379.

[111] M. Van den Bergh, *Calabi-Yau algebras and superpotentials*, Selecta Math. (N.S.) 21 (2015), no. 2. 555-603.

[112] O. van Koert, *Lecture notes on stabilization of contact open books*, arXiv:1012.4359.

[113] C. Viterbo, *Functors and computations in Floer homology with applications*, Part I, Geom. Funct. Anal. 9: 985-1033, 1999.

[114] C. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994.

[115] D. Yang, *The interplay between 2-and-3-Calabi-Yau triangulated categories*, Sci. Sin Math. 48 (2018), 1631-1640.

[116] W. Yeung, *Relative Calabi-Yau completions*, arXiv: 1612.06352.

[117] J. Zhao, *Periodic symplectic cohomologies and obstructions to exact Lagrangian immersions*, PhD thesis, Columbia University, 2016.

[118] Z. Zhou, *Symplectic fillings of asymptotically dynamically convex manifolds I*, arXiv:1907.09510.

[119] Z. Zhou, *Symplectic fillings of asymtotically dynamically convex manifolds II–$k$-dilations*, arXiv:1910.06132.